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HISTORY, BIOGRAPHY

Heller, Siegfried. Ein Fehler in einer Archimedes-Ausgabe, seine Entstehung und seine Folgen. Abh. Bayer. Akad. Wiss. Math. Nat. Kl. (N.F.) no. 63, 39 pp. (4 plates) (1954).

The author calls attention to a mistake which appears in the proof of Theorem 21 in "On conoids and spheroids" by Archimedes. The mistake occurs in vol. I, p. 350, 20–p. 352, 1 in the edition by Heiberg [Archimedis opera omnia, 3 vols., Teubner, Leipzig, 1910, 1913, 1915] and is repeated in the translation by Ver Eecke [Les oeuvres complètes d'Archimède, Desclée, Paris, 1921] on p. 190, lines 15–18, which reads: "Or la somme de ces premières droites est plus grande que le double de la somme de ces secondes droites, exception faite de la droite $A\Delta$." In this statement the words "plus grande" should be replaced by "égale à". Instead of making this correction the author suggests that ΔI [Ver Eecke, p. 190, line 10] be replaced by ΔB, and everything is correct. The author follows this mistake from the oldest manuscripts to the latest translations of the works of Archimedes. The author points out that the mistake is mentioned by H. Hermelink [Arch. Internat. Hist. Sci. (N.S.) **6**, 430–433 (1953); MR **15**, 591] and E. J. Dijksterhuis has also been aware of the mistake in his edition of the works of Archimedes [Euclides, Groningen **15**, 96–112, 113–135 (1938); 248–256, 257–269; **16**, 104–112 (1939); 113–132; **17**, 8–40 (1940); 239–256, 257–265 (1941); **20**, 31–48, 49–74 (1944)]. *O. Schmidt* (Copenhagen).

Ćulum, Živojin. Par quel procédé Archimedes avait-il evalué le résultat: $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$? Bull. Soc. Math. Phys. Serbie **6**, 108–111 (1954). (Serbo-Croatian. French summary)

Alimov, N. G. Quantity and ratio in Euclid. Istor.-Mat. Issled. **8**, 573–619 (1955). (Russian)

Stamatis, E. Über die Irrationalenzahlen bei den Alten. Prakt. Akad. Athēnōn **29** (1954), 337–345 (1955). (Greek. German summary)

★**Burger, D. L'évolution des idées de l'infini de Platon à Cantor.** Actes du VI[e] Congrès International d'Histoire des Sciences, Amsterdam, 1950, pp. 145–150. Hermann & Cie, Paris, 1951.

★**van Wijk, W. E. Un comput de la fin du douzième siècle.** Actes du VI[e] Congrès International d'Histoire des Sciences, Amsterdam, 1950, pp. 133–139 (1 plate). Hermann & Cie, Paris, 1951.

The mathematical treatises of Džemšid Giyasèddin Kaši. Istor.-Mat. Issled. **7**, 9–379 (1954). (Russian)

Yuškevič, A. P., and Rozenfel'd, B. A. Commentary on the mathematical treatises of Džemšid Giyasèddin Kaši. Istor.-Mat. Issled. **7**, 380–449 (1954). (Russian)

These two works together comprise a translation into Russian and a commentary on two books written by Jamshīd Ghiāth al-Dīn al-Kāshānī or al-Kāshī (fl. 1420) who worked in Iran and Turkistan. The publication does not give the Arabic text, but facsimiles of numerous manuscript pages have been included.

The first translation is of "The key to arithmetic" a work already extensively studied by Luckey in two publications [Math. Ann. **120**, 217–274 (1948); Die Rechenkunst bei Ğamšīd b. Mas'ūd al-Kāšī ···, Deutsche Morgenländische Gesellschaft, Steiner, Wiesbaden, 1951; MR **9**, 484; **13**, 611]. Thus the main interest of the Russian translation lies in those sections not dealt with by Luckey. These are in large part a number of worked examples in which the newly invented decimal fractions are freely and adeptly applied. Topics touched on include the mensuration of plane figures, solids, regular and semi-regular polyhedra; layouts and area computations for various arches and other architectural forms; operations with polynomials of arbitrary degrees; diophantine and inheritance problems.

On page 397 a line of characters printed upside down may prove confusing to readers not familiar with the Arabic sexagesimal numerals.

Of the many manuscripts extant (including one in the uncataloged Garrett collection at Princeton), the translator used the Leningrad and Berlin copies. No mention is made of the Tehran lithograph edition of 1889.

The second translation is of the "Treatise on the circumference", a computation yielding an approximation to 2π to ten sexagesimal places. A critical edition of this work with a German translation has recently been published [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1950**, no. 6 (1953); MR **14**, 1051]. *E. S. Kennedy.*

Yuškevič, A. P. On the achievements of Chinese scholars in the field of mathematics. Istor.-Mat. Issled. **8**, 539–572 (1955). (Russian)

Aleksandrov, P. S. Mathematics in the University of Moscow during the first half of the XXth century. Istor.-Mat. Issled. **8**, 9–54 (3 plates) (1955). (Russian)

Golubev, V. V. Mechanics in the University of Moscow before the great October socialist revolution and during the Soviet period. Istor.-Mat. Issled. **8**, 77–126 (2 plates) (1955). (Russian)

Liholetov, I. I., and Yanovskaya, S. A. From the history of the teaching of mathematics in the University of Moscow (1804-1860). Istor.-Mat. Issled. **8**, 127–480 (2 plates) (1955). (Russian)

Tyulina, I. A. The development of mechanics in the University of Moscow in the XVIIIth and XIXth centuries. Istor.-Mat. Issled. **8**, 489–536 (1955). (Russian)

Depman, I. Ya. "Geometriya praktika." Istor.-Mat. Issled. **8**, 620–629 (1955). (Russian)
Discussion of an early Russian book on mathematics.

Buarque de Gusmão, Adauto. History of mathematical analysis. Revista Mineira de Engenharia **18**, no. 65, 19–23 (1955). (Portuguese)

Nowiński, Jerzy. A historical survey of the theory of elasticity. Rozprawy Inż. **3**, 57–78 (1955). (Polish. Russian and English summaries)

Kuratovskiĭ [Kuratowski], K. The scientific activity of the Mathematical Institute of the Polish Academy of Sciences. Uspehi Mat. Nauk (N.S.) **10**, no 3(65), 217–221 (1955). (Russian)

★**A magyar matematika története a XVIII. század végéig.** [**History of Hungarian mathematics from the 18th century to the present.**] Tankönyvkiadó, Budapest, 1953. 40 pp. 250 ft.

★**Vázlatok a magyar matematika újkori történetéből.** [**Outlines of the modern history of Hungarian mathematics.**] Tankönyvkiadó, Budapest, 1953. 68 pp. 450 ft.

Köthe, G. Das Bild der heutigen Mathematik. Experientia **11**, 249–254 (1955).

Infeld, Leopold. Die Geschichte der Relativitätstheorie. Naturwissenschaften **42**, 431–436 (1955).

Myškis, A. D., and Rabinovič, I. M. The first proof of a fixed-point theorem for a continuous mapping of a sphere into itself, given by the Latvian mathematician P. G. Bohl. Uspehi Mat. Nauk. (N.S.) **10**, no. 3(65), 188–192 (1955). (Russian)

Alexits, György. János Bolyai. Mat. Lapok **3**, 107–110 (1952). (Hungarian)

Rényi, Alfréd. János Bolyai, a great revolutionist of science. Mat. Lapok **3**, 173–178 (1952). (Hungarian)

Otradnyh, F. P. V. Ya. Bunyakovskiĭ—professor in the Petersburg university. Vestnik Leningrad. Univ. **10**, no. 5, 49–54 (1955). (Russian)

Dagli scritti di Fabio Conforto. Archimede **6**, 107–122 (1954).
Six papers by Conforto selected for their general interest.

Linnik, Yu. V., and Rodosskiĭ, K. A. Nikolaĭ Grigor'evič Čudakov. (On his fiftieth birthday.) Uspehi Mat. Nauk (N.S.) **10**, no. 3(65), 213–215 (1 plate) (1955). (Russian)
A list of Čudakov's published papers is included.

Sen, N. R. In memoriam: Cuthbert Edmund Cullis. Bull. Calcutta Math. Soc. **46**, 139–140 (1 plate) (1954).

Albert, A. A. Leonard Eugene Dickson, 1874-1954. Bull. Amer. Math. Soc. **61**, 331–345 (1955).
A list of Dickson's published work is included.

Fokker, A. D. Albert Einstein, 14 March 1878-18 April 1955. Nederl. Tijdschr. Natuurk. **21**, 125–129 (1955). (Dutch)

Obituary: Albert Einstein (1879-1955). Ž. Eksper. Teoret. Fiz. **28**, 637–638 (1 plate) (1955). (Russian)

Born, Max. Albert Einstein und das Lichtquantum. Naturwissenschaften **42**, 425–431 (1955).

★**Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XII. Commentationes mechanica ad theoriam corporum fluidorum pertinentes. Vol. prius.** Edidit Clifford Ambrose Truesdell. Societatis Scientiarum Naturalium Helveticae, Lausanne, 1954. cxxv + i + 288 pp. (1 plate). $14.00.

Ce volume renferme les travaux théoriques d'Euler dans le domaine de la mécanique des fluides. Le lecteur y retrouvera des textes demeurés épars dans les Mémoires de l'Académie des Sciences de Berlin et dans les Novi Commentarii Academiae Scientiarum Petropolitanae.

Les quatre premiers mémoires sont universellement connus et l'on peut dire que leur substance — et aussi leur admirable langage scientifique — fait aujourd'hui partie de tout enseignement classique: ils se rapportent aux principes généraux de l'équilibre et du mouvement des fluides et on y trouve, avec la définition explicite de la pression, les équations générales de l'hydrodynamique.

Les autres mémoires concernent la théorie du frottement des fluides, le mouvement non isotherme des fluides et le mouvement des rivières. Moins connus et sans doute moins parfaits que les précédents, ces mémoires offrent cependant un intérêt historique de premier plan et révèlent jusqu'à quel point Euler a poussé l'analyse dans le but de rejoindre les applications, dont il n'a cessé d'avoir le souci depuis la rédaction, en 1738, de sa Scientia navalis. Le dernier de ces mémoires a une curieuse histoire: simple brouillon abandonné par Euler lors de son départ de Berlin et publié sans doute sans l'autorisation de son auteur longtemps après sa rédaction, ce travail constitue le germe de l'hydrodynamique d'Euler, renferme le théorème dit de Bernoulli et offre l'exemple du recours à une variable du type dit de Lagrange.

Ces textes s'accompagnent d'une importante introduction due à C. A. Truesdell. Le savant éditeur ne s'est pas borné à analyser et à commenter avec la plus grande précision chacun des mémoires d'Euler publiés ici. Il met en lumière le lien entre les recherches théoriques d'Euler et ses recherches appliquées et mieux encore situe l'oeuvre d'Euler dans le cadre de son époque, en étudiant l'évolution de l'hydrodynamique de Newton à Lagrange.

On nous permettra, au sujet de cette savante introduction qui renferme une très belle leçon d'histoire, les remarques suivantes: Jugeant la science d'hier avec des yeux modernes, l'éditeur nous révèle en puissance dans l'oeuvre même d'Euler des prolongements tout à fait inattendus; en revanche, il est sans bienveillance à l'égard des tentatives qui n'ont pas survécu. Sa sévérité à l'égard de d'Alembert est extrême: il qualifie son oeuvre de "tortuous and lengthy" et ailleurs de "dense mist of calculation, philosophy and error". Certes, d'Alembert est un auteur difficile et souvent décevant; tous ceux qui l'ont lu en ont fait l'expérience. Certes, son „Traité de l'équilibre et du mouvement des fluides" [David, Paris, 1744] n'a rien apporté de neuf. Certes encore, d'Alembert a été souvent incorrect et même injuste à l'égard de ses

rivaux, et en particulier à l'égard d'Euler. Mais, cela dit, son Essai d'une nouvelle théorie de la résistance de fluides [David, Paris 1752] constitue un véritable travail de pionnier [cf. notre étude: D'Alembert et l'Essai d'une nouvelle théorie sur la résistance des fluides, Bull. Soc. Franç. Méc. no. 6, p. 6ff. (1952)] et à cet égard le jugement de Lagrange conserve toute sa valeur. Il n'est pas exclu que cet Essai, si aride et si imparfait qu'il paraisse aujourd'hui, ait suscité les admirables synthèses d'Euler.

Truesdell nous dit encore qu'Euler a aperçu le premier le paradoxe auquel le nom de d'Alembert est resté attaché. Cela est fort possible; mais Euler n'a pas cessé, grâce à sa foi dans l'analyse mathématique, de nourrir l'espoir d'expliquer la résistance des fluides, tandis que le scepticisme même de d'Alembert le résignait à l'impuissance.

R. Dugas (Paris).

Bodemüller, H. Carl Friedrich Gauss zum Gedächtnis. Z. Vermessungswesen **80**, 33–42 (1955).

An unpublished letter of K. F. Gauss. Vestnik Akad. Nauk SSSR **25**, no. 4, 109–111 (2 plates) (1955). (Russian)

The letter, dated 11 Sept. 1835, was addressed to P. L. Schilling.

★**Cassina, Ugo. Sur les manuscrits et la correspondance de A. Genocchi.** Actes du VI^{e} Congrès International d'Histoire des Sciences, Amsterdam, 1950, pp. 172–177. Hermann & Cie, Paris, 1951.

Maĭstrov, L. E. A. I. Gercen on mathematics. Istor.-Mat. Issled. **8**, 481–488 (1955). (Russian)

Blackwell, David, and Bowker, Albert H. Obituary: Meyer Abraham Girshick, 1908-1955. Ann. Math. Statist. **26**, 365–367 (1955).

Schmeidler, Werner. Zum Gedächtnis an Georg Hamel. Jber. Deutsch. Math. Verein. **58**, Abt. 1, 1–5 (1955).

Gnedenko, B. V. Aleksandr Yakovlevič Hinčin. (On his sixtieth birthday.) Uspehi Mat. Nauk (N.S.) **10**, no. 3(65), 197–212 (1955). (Russian)

A list of Hinčin's published papers is included.

Rényi, Alfréd. On the mathematical work of Károly Jordan. Mat. Lapok **3**, 111–121 (1952). (Hungarian)

A list of Jordan's scientific papers is included.

Keller, Ott-Heinrich, und Engel, Wolfgang. Nekrolog: Heinrich Wilhelm Ewald Jung. Jber. Deutsch. Math. Verein. **58**, Abt. 1, 5–10 (1955).

Aleksandrov, P. S., and Lyapunov, A. A. Lyudmila Vsevolodovna Keldyš (on her fiftieth birthday). Uspehi Mat. Nauk **10**, no. 2(64), 217–223 (1 plate) (1955). (Russian)

A list of her published papers is included.

Two documents concerning N. N. Luzin's biography. Istor.-Mat. Issled. **8**, 55–76 (1 plate) (1955). (Russian)

Künzi, Hans P. Zum 60. Geburtstag von Rolf Nevanlinna. Elem. Math. **10**, 97–100 (1955).

Kravec, T. P., editor. On the selection of Poncelet as a corresponding member of the St. Petersburg Academy of Sciences. Izv. Akad. SSSR. Otd. Tehn. Nauk **1955**, no. 4, 120–130 (1955). (Russian)

Cassina, Ugo. Storia ed analisi del "Formulario completo" di Peano. Boll. Un. Mat. Ital. (3) **10**, 244–265 (1955).

Depman, I. Ya. The first Russian doctor of mathematical sciences of the University of Paris. Istor.-Mat. Issled. **8**, 630–635 (1955). (Russian)

The man in question is P. A. Zateplinskiĭ (1794–?).

FOUNDATIONS, MATHEMATICAL LOGIC

★**Basson, A. H., and O'Connor, D. J. Introduction to symbolic logic.** University Tutorial Press, Ltd., London, 1953. viii+169 pp. 7s. 6d.

An introductory textbook.

★**Gentzen, Gerhard. Recherches sur la déduction logique. (Untersuchungen über das logische Schliessen.)** Traduction et commentaire par R. Feys et J. Ladrière. Presses Universitaires de France, Paris, 1955. xi+170 pp. 1000 francs.

Translation, accompanied by a substantial number of notes, of Gentzen's paper in Math. Z. **39**, 176–210, 405–431 (1934).

Lorenzen, Paul. Die Allgemeingültigkeit der logischen Regeln. Studium Gen. **6**, 605–609 (1953).

This paper, as another one previously reviewed [Actes XIème Congrès Internat. Philos., Bruxelles, 1953, v. 5. North-Holland Publ. Co., Amsterdam, 1953, pp. 12–18; MR **15**, 90], defends an operative interpretation of logic. Logic is concerned with finding out and stating logical rules, that is, metarules which enable us, whenever rules for operations of a certain kind are given, to find new (secondary) rules for these operations. The logical rules are thus meant to simplify operations; this fact explains the universal validity of logical rules. *E. W. Beth.*

Kalinowski, J. Théorie des propositions normatives. Studia Logica **1** (1953), 147–184 (1954). (French. Russian summary)

The formal theory $\mathbf{K}_2$ developed in this paper can be considered to be a theory formalized within a first-order predicate calculus with two types of variables. The first type of variable can be thought to range over subjects of actions and the second over actions. There appears in the theory one primitive relation P (such that "$Px\alpha$" is to be read "x has the right to do α"), primitive one-place predicates $X, Y, \cdots$ for the subject variables and primitive one-place predicates $A, B, \cdots$ for the action variables. The one-place predicates are used to limit the range of their respective variables. There appears further the functor $\mathbf{N}$ of the action variables such that $\mathbf{N}\alpha$ is to be thought to be the action opposite to α, and the functor $\mathfrak{R}$ of the predicates $A, B, \cdots$ which is such that $C\mathfrak{R}A\alpha Rx\alpha \equiv CA\alpha Rx\mathbf{N}\alpha$ must be true for any normative relation R, primitive or defined. After defining the normative re-

lations W, S, L and M which have the meanings for arguments x and α of "x has the right to do α", "x must do α", "x must not do α" and "x can and cannot do α" respectively, the author discusses several syllogisms of Aristotle and two interpretations of $\mathbf{K}_2$, one as a normative logic and one as a modal logic.

The reader should note that the second "C" in $A1$, $A2$ and $A3$ on page 170 should be read for propositions p and q as $KpCpq$. *P. C. Gilmore* (University Park, Pa.).

Kalinowski, Jerzy. Theory of normative propositions. Studia Logica 1 (1953), 113–146 (1954). (Polish)

A translation of the paper reviewed above. *P. C. Gilmore* (University Park, Pa.).

Sobociński, Bolesław. Note on a problem of Paul Bernays. J. Symb. Logic **20**, 109–114 (1955).

This note concerns problem 7 of same J. **18**, 350 (1954). The propositional calculus can be based on either of the axiom systems

(i) $$p\rightarrow\{q\rightarrow[r\rightarrow(s\rightarrow p)]\},$$
$$\neg q\rightarrow[(p\rightarrow q)\rightarrow(p\rightarrow r)],$$
$$(\neg p\rightarrow q)\rightarrow[(p\rightarrow q)\rightarrow q]$$

or

(ii) $$p\rightarrow[q\rightarrow(r\rightarrow p)],$$
$$(p\rightarrow q)\rightarrow[\neg q\rightarrow(p\rightarrow r)],$$
$$(\neg p\rightarrow q)\rightarrow[(p\rightarrow q)\rightarrow q],$$

with modus ponens and substitution as the only rules of inference. Each formula in these axioms contains ≤ 5 occurrences of propositional letters. By means of a suitable matrix the author shows that not all propositional identities can be deduced from the totality of propositional containing ≤ 4 occurrences of propositional letters, with $\rightarrow$ and $\neg$ as logical connectives. The use of & and $\vee$ would not alter the situation since $a\&b$ may be replaced by $\neg(a\rightarrow\neg b)$ and $a\vee b$ by $\neg a\rightarrow b$ without increase of the number of propositional letters. *G. Kreisel.*

Beth, E. W. Remarks on natural deduction. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17**, 322–325 (1955).

The lower predicate calculus operates on two levels, one semantic (model-theoretic), the other deductive, the gap between the two being bridged by the completeness theorem. The author's principal purpose in the present paper is to link up the two levels all along the line. This is achieved by means of certain "semantic tableaux", which can be transformed in a definite way into formal derivations in a system of natural deduction (a "Gentzen calculus"). The author states that the principal metamathematical theorems of the lower predicate calculus can be presented in a natural and elegant fashion within the framework sketched in the paper under review. It is to be hoped that he will publish a more detailed description of his method in due course. *A. Robinson.*

Uspenskiĭ, V. A. Gödel's theorem and the theory of algorithms. Dokl. Akad. Nauk SSSR (N.S.) **91**, 737–740 (1953). (Russian)

The paper considers relations between certain notions of separability of sets of integers on the one hand and incompletability theorems of formal systems on the other. For example, a calculus containing negation and subject to certain restrictions has no consistent decidable extension (i.e. it is essentially undecidable in the sense of Tarski) if and only if there is no recursive set H of formulas, such that H contains all provable formulas and no formulas whose negations are provable. The paper depends of course on an enumeration of formulas of the system and of sets of equations defining partial recursive functions according to Gödel and Kleene. [For further details see the review by Mostowski in J. Symb. Logic **19**, 218–219 (1954). Cf. also Kleene, Nederl. Akad. Wetensch., Proc. **53**, 800–802 (1950); MR **12**, 71.] *H. B. Curry* (University Park, Pa.).

Anderson, Alan Ross. Correction to a paper on modal logic. J. Symb. Logic **20**, 150 (1955).

See same J. **19**, 201–214 (1954); MR **16**, 103.

Kleene, S. C. Hierarchies of number-theoretic predicates. Bull. Amer. Math. Soc. **61**, 193–213 (1955).

This paper (H), together with the papers CO and AP reviewed below, constitutes one of the most important advances in the foundations of mathematics in recent years. It starts with (i) general recursive predicates, followed by (ii) arithmetic predicates $(x)R(x; n)$, $(x)(Ey)R(x, y; n), \cdots$, $(Ex)S(x; n)$, $(Ex)(y)S(x, y; n), \cdots$, where R and S are in (i), then (iii) hyperarithmetic predicates described in AP, and, finally, (iv) analytic predicates $(\alpha)(Ex)R(\alpha, x; n)$, $(\alpha)(E\beta)(x)R[\alpha, \beta, x; n], \cdots$, $(E\alpha)(x)S(\alpha, x; n)$, $(E\alpha)(\beta)(Ex)S(\alpha, \beta, x; n)\cdots$; here α and β are intended to range over (arbitrary) functions whose arguments and values are integral, and R and S are built up from the number variables x and n, the function variables a, β, $\cdots$ and symbols for general recursive functions and predicates (cf. AP). The author reviews known results concerning (i) and (ii), and proves: if $P(n)$ can be expressed in both one-function-quantifier-forms, i.e. $\leftrightarrow(\alpha)(Ex)R(\alpha, x; n)\leftrightarrow$ $(E\beta)(x)S(\beta, x; n)$, then $P(n)$ is in (iii). This is the analogue to the author's theorem that a predicate which can be expressed in both one-(number) quantifier-forms of (ii) is in (i). More generally, the predicates of (iv) are, in most respects, related to those of (iii) as (ii) is related to (i). Accordingly, in analogy to the degrees of predicates P and Q [Kleene and Post, Ann. of Math. (2) **59**, 379–407 (1954); MR **15**, 772], where $\deg P\leq\deg Q$ if P is recursive in Q, the author introduces a hyper degree where hyp deg $P\leq$hyp deg Q if P is hyperarithmetical in Q. Several analogues of known results about the degrees of arithmetic predicates are established, and open problems pointed out.

The proof of the main theorem yields a general recursive R such that $(E\alpha)(x)R(\alpha, x)$ holds, yet for each hyperarithmetic β we have $(Ex)\neg R(\beta, x)$. The proof of $(E\alpha)(x)R(\alpha, x)$ uses a formal enumeration of (iii) by means of a one-function-quantifier-predicate constructed in CO.

In the reviewer's opinion the class (iii) provides a precise and satisfactory definition of the notion of predicative sets (of integers), based on the concept of constructive ordinal. The present paper shows that there are propositions of the simple form $(E\alpha)(x)R(\alpha, x)$ which can be proved in quite weak impredicative set theories, but are not satisfied by any (constructively) predicative set α. *G. Kreisel* (Princeton, N.J.).

Kleene, S. C. Arithmetical predicates and function quantifiers. Trans. Amer. Math. Soc. **79**, 312–340 (1955).

The most general predicates obtainable by quantification over the integers and arbitrary functions of

integers are described, and two sets of normal forms are established; one for unbounded functions α and one for α^* whose values are 0 and 1; here α is represented by α^* as follows: if $\alpha^*(0)=1$ and $\alpha^*(n)=1\rightarrow\alpha^*(n+1)=0$, then the sequence $\alpha^*(0)$, $\alpha^*(1),\cdots$ consists of stretches of 0 separated by isolated 1, and the length of the nth stretch of zeros represents $\alpha(n)+1$. In analogy with arithmetic predicates the prefixes of normal forms are partially ordered by the number of alternating quantifiers, and if a prefix Π does not precede Π' then the class definable by Π is not included in the class definable by Π'. Next the author considers an analogue to Post's theorem [Bull. Amer. Math. Soc. **54**, 641–642 (1948)]: if an arithmetic predicate is expressible in both $(k+1)$-quantifier-forms, it is general recursive in a k-quantifier-form. This is false if 'arithmetic' is replaced by 'analytic' and 'general recursive' by 'arithmetic': the result suggests the analogue obtained when 'general recursive' is replaced by 'hyperarithmetic' as is done in H reviewed above. The hyperarithmetic predicates consist of a set of predicates $T_n(a)$ labelled by constructive ordinals (represented by the integer) n: if n is the successor of m, $T_n(a)$ is obtained by means of quantification of T_m, if n is the limit of the ascending sequence $m(p)$, $p=1, 2,\cdots$, $T_n(a)$ is obtained by enumeration of the predicates $T_{m(p)}(a)$. If the predicates T_n are classified according to the ordinal number $|n|$ represented by n, they form a hierarchy because, if $|n|=|m|$, then $\deg T_n=\deg T_m$, and, if $|n|<|m|$, $\deg T_n<\deg T_m$. As a digression the places of the predicates introduced by Wang [J. Symb. Logic **18**, 49–59 (1953); MR **14**, 936] and Myhill [ibid. **18**, 115–118 (1953); MR **15**, 924] in the hierarchy are determined. Finally, Mostowski's hyperarithmetic hierarchy [Studia Philos. **4**, 237–274 (1951)] is described carefully and put in 1-1 correspondence with the author's.

Reviewer's note. The author remarks that the distinctness of the classes of the arithmetic hierarchy is proved informally and classically: a formal proof from the assumption of (a suitable arithmetic version of) the consistency of arithmetic is given by the reviewer [British J. Philos. Sci. **3**, 364–374 (1953), p. 370].

G. Kreisel (Princeton, N.J.).

Kleene, S. C. On the forms of the predicates in the theory of constructive ordinals. II. Amer. J. Math. **77**, 405–428 (1955).

The author corrects and continues the investigations of a previous paper with the same title [same J. **66**, 41–58 (1944); MR **5**, 197]. He gives a condensed account of constructive versions for Cantor's two approaches to the theory of ordinal numbers, namely order types of well-orderings, and entities generated by forming a successor and taking the limit of increasing sequences: 'constructive' well-orderings are taken to be (primitive) recursive, and in a 'constructive' generation the sequences considered are recursive; they are represented by an integer as described in Church and Kleene, Fund. Math. **28** 11–21, (1937). Markwald [Math. Ann. **127**, 135–149 (1954); MR **15**, 771] has shown that the same ordinal numbers are obtained by means of these two approaches. The principal result of the present paper is this: if 0 is the class of integers which represent constructive ordinals then, for every general recursive R, a primitive recursive function $\xi_R(n)$ can be effectively determined such that $(\alpha)(Ex)R(\alpha, x, n)\leftrightarrow \xi_R(n)\in 0$. The proof is suggested by a construction due to Brouwer [ibid. **97**, 60–75 (1926)] who associates with a given relation R an ordering S of finite sequences of integers such that S is a well-ordering if and only if $(\alpha)(Ex)R(\alpha, x)$. The author goes on to construct a recursive R_0 such that $(\alpha)(Ex)R_0(\alpha, x; n)\leftrightarrow n\in 0$; this is very plausible from the approach to ordinals by way of recursive orderings $<$, since $(\alpha)(Ex)[\alpha(x+1)<\alpha(x)]$ means that $<$ is a well-ordering. By use of AP it is easily shown that $n\in 0$ cannot be expressed in the form $(E\alpha)(x)S(\alpha, x; n)$; in particular, it is not arithmetical since

$$(x)(Ey_1)\cdots(x_n)(Ey_n)R(x_1\cdots x_n, y_1\cdots y_n)\leftrightarrow$$
$$(E\alpha_1)\cdots(E\alpha_n)(x_1)\cdots(x_n)R[x_1\cdots x_n\alpha_1(x_1)\cdots\alpha_n(x_1\cdots x_n)],$$

and also

$$\leftrightarrow(x_1)(\beta_2\cdots(\beta_n)(Ey_1)\cdots$$
$$(Ey_n)R[x_1\beta_1(y_1)\cdots\beta_n(y_1\cdots y_{n-1})y_1\cdots y_n].$$

From the last result mentioned in the preceding review, the author obtains a recursive ordering of the integers which is not well-ordered, but all descending hyperarithmetic sequences are finite. This shows that it may be necessary to consider impredicatively defined sequences in order to establish that a given integer does not represent a constructive ordinal. For the purposes of a 'constructive' theory of ordinals it would be of interest to consider those integers n which do represent constructive ordinals, i.e. are generated by means of 01-04 on p. 408, and investigate for such n whether a suitably chosen finite number of (true) arithmetic propositions suffice to express the fact that n is generated by means of 01-04.

G. Kreisel (Princeton, N.J.).

Löb, M. H. Solution of a problem of Leon Henkin. J. Symb. Logic **20**, 115–118 (1955).

Let (S) be a formal system containing recursive arithmetic and Gödel's substitution function for a suitable Gödel numbering of (S), $\{A\}$ denote the Gödel number of the formula A, $\mathfrak{P}(a)$ be a predicate of (S) which is intended to express 'provability in (S) of the formula with number a' and which satisfies the following conditions for any A and B of (S):

(i) $\quad \vdash_S\mathfrak{P}(\{A\rightarrow B\})\rightarrow[\mathfrak{P}(\{A\})\rightarrow\mathfrak{P}(\{B\})]$,

(ii) if A is provable in (S), $\vdash_S\mathfrak{P}(\{A\})$,

(iii) $\quad \vdash_S\mathfrak{P}(\{A\})\rightarrow\mathfrak{P}(\{\mathfrak{P}(\{A\})\})$.

Then, if $\vdash_S\mathfrak{P}(\{A\})\rightarrow A$ also $\vdash_S A$. (The author has 5 conditions, but II is not used, and III only for the proof of V.)

The conditions (i)-(iii) are satisfied by the usual proof predicate for Peano's arithmetic, even if '$\{A\}$' and '$\{B\}$' are replaced by free variables. The author's ingenious method of proof is related to an informal paradox discovered by P. T. Geach [Analysis **15**, 71–72 (1955)]. Under the present conditions on $\mathfrak{P}$, the result leads immediately to a positive solution of problem 3 in J. Symb. Logic **17**, 160 (1952): is $\vdash_S A$ if A is the formula $\mathfrak{P}(\{A\})$, i.e. 'says' of itself that it is provable in (S)?

Reviewer's notes. Condition (i) cannot be replaced by the weaker condition (i'): if $\vdash_S\mathfrak{P}(\{A\rightarrow B\})$, then $\mathfrak{P}(\{A\})\rightarrow\mathfrak{P}(\{B\})$ is true; for, the proof predicate $\mathfrak{P}_1$ of the reviewer's note [Nederl. Akad. Wetensch. Proc. Ser. A. **56**, 405–406 (1953), p. 406; MR **15**, 668] satisfies (i'), (ii) and (iii) though, for this $\mathfrak{P}_1$, Henkin's problem has a negative solution. Condition (i) is not satisfied for this $\mathfrak{P}_1$, if (S) is Peano's arithmetic Z: for any A,

$$\vdash_Z\mathfrak{P}_1(\mathfrak{q}_1)\&A\cdot\rightarrow\mathfrak{P}_1(\mathfrak{q}_1)$$

and $\vdash_Z\sim\mathfrak{P}_1(\mathfrak{q}_1)$, so, on (i),

$$\vdash_Z\sim\mathfrak{P}_1(\{\mathfrak{P}_1(\mathfrak{q}_1)\&A\});$$

but if $\mathfrak{P}$ is the usual proof predicate for Z,

$$\vdash_Z \sim\mathfrak{P}_1(\{\mathfrak{P}_1(q_1)\&A\}) \leftrightarrow \sim\mathfrak{P}(\{\mathfrak{P}_1(q_1)\&A\})$$

since $\{\mathfrak{P}_1(q_1)\&A\} \neq q_1$, and $\vdash_Z \sim\mathfrak{P}(\{\mathfrak{P}_1(q_1)\&A\})$ is impossible since $\mathfrak{P}$ satisfies the conditions for Gödel's second undecidability theorem [Hilbert and Bernays, Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, p. 285]. For comparison with Gödel's incompleteness theorem: if A_1 is the formula $\sim\mathfrak{P}(\{A_1\})$ and (S) is consistent, then $\vdash_S A_1$ is false provided only (ii) is satisfied. In contrast to this, for the author's result it is essential that $\mathfrak{P}$ not only possesses the properties which are expected of any 'provability' predicate, but that they are formally deducible in (S) itself. *G. Kreisel.*

Halphen, Etienne. La notion de vraisemblance. Publ. Inst. Statist. Univ. Paris **4**, 41–92 (1955).

This paper, written in 1947 and now published posthumously, is an essay on the foundations of statistics. The author's view of statistics is subjectivistic or personalistic, and accordingly revolves around ideas of utility and personal probability. His general ideas on these subjects are not very different in most respects from ideas that had been expressed by others (of whom he seems to have been unaware) by 1947. But synthesis of the two ideas was then presumably unheard of, except for F. P. Ramsey's papers of 1926 and 1928 [reprinted in "The foundations of mathematics and other logical essays," Paul, Trench, Trubner, London, 1931]; and application of this synthesis to statistical inference was then altogether new. The paper has much new material of worth even today, more philosophical than mathematical, in its many aphoristic sidelights and insights. *L. J. Savage.*

Vermeulen, R. Validity of hypotheses. Synthese **9**, 385–394 (1955).

An exposition of the justification of scientific induction in terms of inverse probability, as given in section 1.6 of H. Jeffreys, "Theory of probability" [Oxford, 1939; MR **1**, 151]. If $p_1, p_2, \cdots$ form a sequence of consequences of a hypothesis H, and these consequences are successively found to be true, then $P(p_n|hp_1p_2\cdots p_{n-1}) \to 1$ as $n \to \infty$ even if H is false. Here h is a given proposition, and the expression to the right of the vertical stroke represents a logical conjunction. The philosophy differs from that of Jeffreys in being more sympathetic to the subjectivistic than to the necessary interpretation of probability (as a unique rational degree of belief). *I. J. Good.*

ALGEBRA

Linear Algebra, Polynomials, Invariants

★**Kuroš, A. G. Kurs vyššeĭ algebry. [A course of higher algebra.] 4th ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 379 pp. 8.30 rubles.

For a review of the 2d edition [1949] see MR **12**, 73. Although this edition has been revised, the plan and contents are nearly the same.

Andreoli, Giulio. Operazioni binarie. Scale di operazioni. Ricerca, Napoli **5**, no. 3, 3–11 (1954).

The paper begins with some observations (in some respects misleading) about operations of binary composition on a set E, which lead the author to consider the following case. E is the complex domain and one defines, for a, x in E, $a(+)x = \phi^{-1}(\phi(a)+\phi(x))$, calling this the transform of addition by the function ϕ. The set of operations, all commutative and associative, obtained by iterating ϕ positively and negatively is called a scale. With $\phi = \log$, $\phi^{-1} = \exp$, each operation is distributive over the next below it in the scale. This situation is generalised to the case when E consists of infinite sequences of complex numbers, the corresponding elements of two sequences being combined by different operations of the previous logarithmic scale. Scales of such operations on sequences are constructed; again each operation of a scale is distributive over its predecessor. One can have finite scales. Possibilities of further generalisation are briefly explored. *I. M. H. Etherington.*

Shoda, Kenjiro. Bemerkungen über die Existenz der algebraisch abgeschlossenen Erweiterung. Proc. Japan Acad. **31**, 128–130 (1955).

The present paper is an addition to a previous paper [Osaka Math. J. **4**, 133–143 (1952); MR **14**, 614]. It is shown that the conditions for the existence of algebraically closed extensions of a general algebraic system may be formulated in a somewhat different manner.

O. Ore. (New Haven, Conn).

Stojaković, Mirko. Une théorie générale axiomatique des déterminants. Bull. Soc. Math. Phys. Serbie **6**, 40–55 (1954). (Serbo-Croatian summary)

A slight variation of K. Hensel's definition of a determinant [J. Reine Angew. Math. **159**, 246–254 (1928)] is given. An analogous definition of the "determinant" of an l by k matrix A with $l \leq k$ (= the sum of all the l-rowed minors of A) is then conjectured but not proved. The remainder of the paper repeats earlier work of the author [Bull. Soc. Math. Phys. Serbie **4**, nos. 1–2, 9–23 (1952); MR **14**, 443]. (Reviewer's comment: The author's assertion that the sum $\varphi_n(A)$ of all n-rowed minors of A is a multiplicative function is incorrect when $n < l$.)

M. F. Smiley (Iowa City, Iowa).

Torroja, J. M. Calculation with cracovians. Univ. Madrid Fac. Ci. Sem. Astr. Geodes. Publ. no. **21**, 9 pp. (1953). (Spanish)

This is an exposition of the elementary algebra of cracovians, which differ from matrices in that multiplication is column-by-column.

G. E. Forsythe (New York, N.Y.).

Derwidué, L. Sur la résolution des systèmes d'équations algébriques. Bull. Soc. Roy. Sci. Liège **24**, 172–188 (1955).

La résolution d'un système général d'équations algébriques est un problème bien connu, surtout grâce aux travaux de Kronecker, Halphen, Severi, van der Waerden et A. Weil. Néanmoins, il n'en existe guère d'exposé à la fois accessible aux étudiants de nos candidatures en sciences et suffisamment complet pour servir de base à des études ultérieures approfondies. L'objet de cette note est de presénter un tel exposé. Pour des raisons de simplicité, on se limitera au champ complexe.

Extract from the paper.

Hopkins, A. Olive. Concomitants of the quintic of degree five in the coefficients of the ground form. J. London Math. Soc. **30**, 319–324 (1955).

This paper concerns the calculation of $\{5\}\otimes\{5\}$ which gives the types of all the concomitants of degree 5 in the coefficients of a quintic in any number of variables. The method employed uses a tentative technique of the reviewer [Philos. Trans. Roy. Soc. London. Ser. A **239**, 305–365 (1944), p. 349; MR **6**, 41], together with operational methods due to H. O. Foulkes [ibid. **246**, 555–591 (1954); MR **15**, 926]. There is reference also to $\{5\}\otimes\{\mu\}$ for partitions (μ) of 5. *D. E. Littlewood* (Bangor, Wales).

See also: Cohn, p. 88; Markovitch; p. 88; Stiefel, p. 88; Shanks, p. 88.

Lattices

Morinaga, Kakutaro, and Nishigôri, Noboru. A note on correction. J. Sci. Hiroshima Univ. Ser. A. **18**, 399–400 (1955).

See same J. **16**, 177–221, 399–408 (1952); MR **15**, 389.

Dwinger, Ph. The closure operators of the cardinal and ordinal sums and products of partially ordered sets and closed lattices. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17**, 341–351 (1955).

Continuing his earlier work [same Proc. **58**, 36–40 (1955); MR **16**, 668], the author considers closure operators on various combinations of partially ordered (p.o.) sets. If X is the cardinal sum of p.o. sets Y and Z, then the set C_X of all closure operators on X is isomorphic to $C_Y C_Z$ and to a subset of C_{YZ}; if Y and Z have zeros, there is an isotone mapping of C_{YZ} onto $C_Y C_Z$. If Y and Z are closed lattices, then C_{YZ} and $C_Y C_Z$ are completely meet-homomorphic, and $C_Y C_Z$ is isomorphic to a closed sublattice of C_{YZ}. If X is the ordinal sum of p. o. sets Y and Z, then there is an isotone mapping of $C_{Y\oplus Z}$ onto C_Z; if Y and Z are complete lattices, this mapping is a complete lattice-homomorphism. For the ordinal product X of p. o. sets Y and Z, there is an isotone mapping of C_X onto C_Y; the case of closed lattices was covered in the paper cited above. *P. M. Whitman* (Silver Spring, Md.).

Yamamoto, Koichi. Note on the order of free distributive lattices. Sci. Rep. Kanazawa Univ. **2**, no. 1, 5–6 (1953).

Let $f(n)$ denote the number of elements in the free distributive lattice with n generators. By considering involutions, the author shows that if n is even then $f(n)$ is even. He conjectures that for all n, $f(n)$ is divisible by every prime divisor of $n+2$. *P. M. Whitman.*

Benado, M. La normalité d'Ouzkov et le théorème Jordan-Hölder. Com. Acad. R. P. Române **1**, 7–11 (1951). (Romanian. Russian and French summaries)

A modified form of Uzkov's conditions [Mat. Sb. N.S. **4(46)**, 31–43 (1938)] is shown, in outline, equivalent to the Schreier-Zassenhaus theorem for algebras. The interrelations of several forms of normality are considered briefly. [Cf. the author's other papers, such as Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1**, 282–317 (1951); **4**, 69–120 (1953); MR **16**, 212.] *P. M. Whitman.*

Benado, Milhail. Sur le théorème de raffinement de O. Schreier. Com. Acad. R. P. Române **1**, 1021–1023 (1951). (Romanian. Russian and French summaries)

The Zassenhaus refinement is investigated for β-normal chains [cf. the references in the preceding review]. *P. M. Whitman* (Silver Spring, Md.).

See also: Hall, p. 73.

Rings, Fields, Algebras

Almeida Costa, A. On modules and rings with operators. Univ. Lisboa. Rev. Fac. Ci. A. (2) **4**, 5–62 (1955).

This is largely an expository work dealing with primitive rings, simple rings (not necessarily associative) and their modules. A highlight is a careful analysis of the extent to which concepts vary with the operator domain. For example: a simple ring with operators is also simple in the absolute sense. *I. Kaplansky* (Chicago, Ill.).

Tiago de Oliveira, J. Normal basis and semi-simple modules. Univ. Lisboa. Revista Fac. Ci. A. (2) **4**, 263–272 (1955).

Let R be an arbitrary ring with unit contained in a ring S with the same unit. Then if M is any unitary right R-module, the author first defines the Kronecker product $M\otimes_R S$ and gives several of its elementary properties. There are several misleading statements in this part of the paper. For example, it is stated that if $M\otimes_R s=0$ then $s=0$ (Th. 9), which need not be true. In the second part of the paper the author proves several elementary theorems about completely reducible modules. One half of Th. 12 is again not correct: it is well known that a completely reducible module need have no basis. *A. Rosenberg* (Princeton, N.J.).

Claus, Heinz Jörg. Über die Partialbruchzerlegung in nicht notwendig kommutativen euklidischen Ringen. J. Reine Angew. Math. **194**, 88–100 (1955).

This is an extension of a theorem of Ostmann [same J. **188**, 150–161 (1950); MR **12**, 668] to the case of non-commutative rings. The ring E in question is assumed to have no zero divisors and to satisfy a right division algorithm with respect to a positive (essentially integer-valued) value function w such that $w(a)<w(b)\leftrightarrow w(ac)<w(bc)$ and $w(ab)=w(ba)$. This division algorithm implies a unique factorization theorem in E, the existence of g. c. d.'s and l. c. m.'s and the embeddability of E in a division ring of quotients. Every pair (a, b) of elements in E and every factorization $b=\prod b_i$ of b into relatively prime factors b_i give rise to a partial fraction decomposition (of ab^{-1}): $a=\sum R_i b_i B_i+\sum r_i B_i$ with $B_i=\prod_{j\neq i} b_j$ and $w(r_i)<w(b_i)$. (Also, factorization of b into prime powers gives rise to a partial fraction decomposition similarly analogous to the usual one.) The first sum is called the integral part of the decomposition and the second sum the fractional part. If the partial fraction decomposition of every pair is uniquely determined once the integral part is given, then the ring E is the ring of formal polynomials over a division ring K in one variable x which semi-commutes with K in the sense that $xk=\bar{k}x+k'$. If both the integral and fractional part are uniquely determined in the partial fraction decomposition, then either $E=K$, or $E=K[x]$ with K a field and x an indeterminate commuting with K. *D. Zelinsky* (Kyoto).

Levitzki, J. Some theorems concerning associative Zorn rings. Bull. Res. Council Israel 3, 380–384 (1954).

The author continues his study of the rings of the title; those in which every non-nil right ideal contains an idempotent not zero. Herein he establishes for matrix extensions of Zorn rings without nilpotents (plain rings) some of the properties of matrix extensions of division rings; for example, (i) if T is a plain ring the index of nilpotency of the matrix ring T_n is n, (ii) if U is a nil-subring of T_n then $U^n=0$, (iii) if $T_n \subset T_m$ and contains the identity then n divides m. *W. G. Lister.*

Hion, Ya. V. Ordered associative rings. Dokl. Akad. Nauk SSSR (N.S.) **101**, 1005–1007 (1955). (Russian)

Let R be a (simply) ordered associative ring. Call x infinitely small (large) if $nx^2 \leq |x|$ ($n|x|<x^2$) for every positive integer n, finite if it is neither. Let K be the set of all finite or infinitely small elements, M the subset of all infinitely small elements; K is a subring of R, M is a convex two-sided ideal in K, and K/M is order-isomorphic to a subring of the real numbers. Various aspects of the ideal theory and structure of R and K are studied. The following result is noteworthy: if R has divisors of 0, then it has nilpotent ideals. As a consequence one obtains that a nil-ordered ring is locally nilpotent. Proofs of some of the thirteen theorems are briefly sketched. *I. Kaplansky* (Chicago, Ill.).

Andreian, C. Le principe du maximum de Zorn dans la théorie des anneaux sans l'hypothèse des chaînes de diviseurs. Com. Acad. R. P. Române **1**, 319–322 (1951). (Romanian. Russian and French summaries)

The author applies Zorn's lemma to obtain the existence of ideals (multiplicative systems) maximal with respect to exclusion of a given set (ideal) in a ring [cf. pp. 104–5 of McCoy, "Rings and ideals", Open Court, La Salle, Ill., 1948; MR **10**, 96]. As a first application the author then shows the existence of minimal and maximal prime ideals belonging to a given ideal in an arbitrary commutative ring; a result due to Krull [Math. Ann. **101**, 729–744 (1929)], and proved here as in McCoy (loc. cit.). Secondly a few basic properties of the quasi-primary ideals defined by L. Fuchs [Acta Univ. Szeged. Sect. Sci. Math. **11**, 174–183 (1947); MR **9**, 77] are demonstrated for a commutative ring with no chain-condition. *A. Rosenberg* (Princeton, N. J.).

Fuchs, L. On a new type of radical. Acta Sci. Math. Szeged **16**, 43–53 (1955).

An element a of an arbitrary associative ring R is an l-zerofactor (=left zerofactor) if there is a $b\neq 0$ in R such that $ab=0$. A nonvoid subset A of R is an l-zerofactor if each element of A is an l-zerofactor. An ideal A is an l-zeroid (=left zeroid) ideal if $A+B$ is an l-zerofactor whenever B is an l-zerofactor. The join $Z^{(l)}$ of all l-zeroid ideals is again an l-zeroid ideal, and is termed the left radical of R. Similarly, by changing the roles of left and right zerofactors one defines the right radical $Z^{(r)}$ of R. The intersection $Z=Z^{(l)}\cap Z^{(r)}$ is termed the zeroid radical of R. By Zorn's lemma there exist maximal l-zerofactor ideals. These ideals are prime, and their intersection is equal to $Z^{(l)}$. A similar result holds for $Z^{(r)}$. Thus Z is just the intersection of all maximal l-zerofactor ideals and all maximal r-zerofactor ideals. While the difference ring $R-Z$ has no nonzero nil ideals, it is not known whether or not its zeroid radical is equal to zero. The zeroid radical Z contains the maximal nilideal of the ring, whereas its intersection with the Jacobson radical J is in general a proper subset of Z as well as of J. In a ring with unit element and with minimum condition on right ideals, Z coincides with the join of all nilpotent ideals. If every finite subset of a l-zerofactor (r-zerofactor) ideal has a right (left) nonzero annihilator, then the zeroid radical of the ring of all $n\times n$ matrices over R is the ring of all $n\times n$ matrices over Z. If a commutative algebra A over a field φ contains at least one element which is not a zerofactor, then the elements of the zeroid radical are either nilpotent or transcendental over φ. The zeroid radical of a subdirectly irreducible commutative ring consists of the set of all zerofactors. *J. Levitzki* (Jerusalem).

Tominaga, Hisao. A note on matrix rings. Math. J. Okayama Univ. **4**, 189–191 (1955).

For any ring R denote by R_n the total $n\times n$ matrix ring over R. A ring R is said to be m-irreducible if R is not representable as a total $n\times n$ matrix ring over any ring with $n>1$. A ring R is called weakly semi-primary if for the Jacobson radical N the quotient ring R/N is a finite or an infinite complete direct sum of finite total matrix rings over division rings: $R/N=\sum_\sigma^C (D_\sigma)_{n_\sigma}$. If, in addition, N is nil then R is called semi-primary. It is shown that if $R=(R_1)_n=(R_2)_n$, R has an identity and R is weakly semi-primary, then $R_1\cong R_2$. With the help of this result the following theorem is proved: If R is semi-primary with identity and n is the greatest common divisor of the n_σ in the above-mentioned representation of R/N, then R is a total $n\times n$ matrix ring over an m-irreducible ring, and if $R=(R_1)_{n_1}=(R_2)_{n_2}$ for m-irreducible rings R_1 and R_2, then $n_1=n_2=n$ and $R_1\cong R_2$. *J. Levitzki* (Jerusalem).

Hua, Loo-Kong. A generalization of Hamiltonian matrices. Acta Math. Sinica **3**, 12–58 (1953). (Chinese. English summary)

An English version has appeared in Acta Sci. Sinica **2**, 1–58 (1953); MR **16**, 6.

Divinsky, Nathan. Pseudo-regularity. Canad. J. Math. **7**, 401–410 (1955).

An element x of a ring A is said to be right-pseudo-regular (r.p.r.) of degree n if for some $y\in A$ one has $x+x^ny+x^{n+1}y=0$. For $n=0$ one gets the familiar right quasi-regularity (r.q.r.). If x is r.p.r. of degree n, x is r.p.r. of degree $n-1$, and thus x is r.q.r. For $n>1$, right-pseudo-regularity of degree n is equivalent to that of degree 2. The join of all r.p.r. right ideals of degree n (i.e. right ideals whose elements are r.p.r. of degree n) is a maximal two-sided r.p.r. ideal R_n of degree n. The author shows, however, that $R_n=0$ for $n>1$. He sets $R_1=R$ and terms this ideal the right subradical of A. Similarly, starting with the notion of left pseudo-regularity he defines the left subradical L of A. An example is given where $R\neq L$. In general R and L are proper subsets of the Jacobson radical J. However, if e.g. $x\in xA$ for every x, then $J=R$. The subradicals enjoy certain radical-like properties. Thus e.g. $A-R$ has zero right subradical, and the subradical $R(A_n)$ of a total $n\times n$ matrix ring A_n is a total $n\times n$ matrix ring R_n over R. The effect of certain restrictions such as the descending chain condition or commutativity upon the relationship between R and J is discussed and some problems are posed. His main result for the commutative case is: If A is a commutative ring whose subradical R is zero, then A is isomorphic to a subdirect sum of subdirectly

irreducible rings $A_1, A_2, \cdots, B_1, B_2, \cdots$, where the A_i are fields and the B_i are bound to their maximal nilideals in the sense of M. Hall. If A satisfies either the descending or the ascending chain condition, then the B_i are nilpotent. *J. Levitzki* (Jerusalem).

Jacobson, N. A note on two dimensional division ring extensions. Amer. J. Math. 77, 593–599 (1955).

For a division ring Δ and a division subring Γ denote by $[\Delta:\Gamma]_L$ (respectively by $[\Delta:\Gamma]_R$) the left (respectively the right) dimension of Δ over Γ. Let Φ and Ψ be the centres of Δ and Γ. The author proves first the following theorem concerning dimensionalities: (1) If $[\Delta:\Gamma]_L<\infty$ and $[\Gamma:\Psi]<\infty$ then $[\Delta:\Phi]<\infty$. (2) If $[\Delta:\Phi]<\infty$ then $[\Gamma:\Psi]\leq[\Delta:\Phi]$, the equality holding if and only if the division subring $\Phi(\Gamma)$ generated by Φ and Γ coincides with Δ. The division ring Γ is said to be Galois in Δ if Γ is the set of invariants of a group of automorphisms acting in Δ. The author shows that if $[\Gamma:\Psi]<\infty$, $[\Delta:\Gamma]_L=2$ and the characteristic is $\neq 2$, then Γ is Galois in Δ. The aim of the remaining part of the paper is to show that if in this theorem the condition $[\Gamma:\Psi]<\infty$ (which by the result on dimensionalities is equivalent with $[\Gamma:\Phi]<\infty$) is dropped, then Γ need not be Galois. Let M be a vector space over a field Φ and F the free algebra $F=\Phi\cdot 1\oplus M\oplus(M\otimes M)\oplus(M\otimes M\otimes M)\oplus\cdots$ based on the space M. If $Q(x)$ is a quadratic form on M and B the ideal in F generated by $x\otimes x-Q(x)\cdot 1$, then $\Delta(M,Q)=F/B$ is called a Clifford algebra of M relative to Q. If Q is nondegenerate, then $\Delta(M,Q)$ is central simple, and if in addition dim $M=\aleph_0$, then $\Delta(M,Q)$ is a Kronecker product of a countable number of generalized quaternion algebras. Conversely, any such product can be obtained from a suitable Clifford algebra. This result implies that there exist Clifford algebras which are division algebras with dim $M=\aleph_0$. Now let $\Delta=\Delta(M,Q)$ be any Clifford division algebra and H a hyperplane in M, i.e. $M=H\oplus\Phi u$ where $0\neq u\in M$. Let $\Gamma=\Delta(H,Q)$ be the subalgebra generated by 1 and H. Then Γ is a division algebra and $[\Delta:\Gamma]_L=[\Delta:\Gamma]_R=2$. In case the characteristic is $\neq 2$, then Γ is Galois in Δ if and only if $H^2\neq(0)$. Now by starting with an $\aleph_0$-dimensional vector space M over a field with characteristic $\neq 2$, one can find hyperplanes H with $H^2=(0)$. Hence it follows by these results that there exist infinite-dimensional Clifford division algebras with characteristic $\neq 2$ which are two-dimensional non-Galois extensions of division algebras. *J. Levitzki* (Jerusalem).

Reiner, Irving. Maximal sets of involutions. Trans. Amer. Math. Soc. 79, 459–476 (1955).

Soit E un module libre de rang n sur l'anneau des entiers Z, et soit U_n le groupe des automorphismes (nécessairement unimodulaires) de E. L'auteur étudie les involutions u dans U_n; en plongeant E dans un espace vectoriel de dimension n sur le corps des rationnels, on voit que u est déterminé par deux sous-modules W^+, W^- de E, de rangs p, q respectivement, d'intersection 0, tels que $u(x)=x$ dans W^+, $u(x)=-x$ dans W^-, et que $p+q=n$. La somme $N=W^++W^-$ n'est pas nécessairement E tout entier, mais les diviseurs élémentaires de N par rapport à E sont égaux à 1 ou à 2; si r est le nombre de ceux qui sont égaux à 2, l'auteur montre que les trois nombres p, q, r caractérisent les éléments d'une classe d'involutions conjuguées dans U_n, et retrouve la forme normale des involutions d'une même classe qu'il avait donnée dans un travail antérieur en collaboration avec L. K. Hua [Trans. Amer. Math. Soc. 71, 331–348 (1951); MR 13, 328]. Il étudie ensuite les ensembles maximaux d'involutions permutables deux à deux dans U_n, et plus particulièrement ceux qui ont le plus grand nombre possible 2^n d'éléments; deux tels ensembles maximaux ne sont pas nécessairement conjugués dans U_n; l'auteur donne un procédé permettant pour chaque n de former toutes les classes de tels systèmes, mais il ne semble pas aisé d'en calculer le nombre de façon générale. Il n'est pas facile non plus de calculer le nombre maximum $f(p,q,r)$ des involutions de type (p,q,r) contenues dans un ensemble maximal de 2^n involutions conjuguées. Toutefois, l'auteur prouve que pour $n>4$, on a $f(p,q,r)>n$ sauf lorsque les involutions sont de type $(n-1,1,0)$, $(n-1,1,1)$, $(n,0,0)$ ou les types obtenus en intervertissant p et q. En outre, on peut distinguer les involutions de type $(n-1,1,0)$ de celles de type $(n-1,1,1)$ (pour $n>2$) en observant que tous les systèmes maximaux contenant n involutions du premier type sont conjugués, alors que c'est inexact pour le second type. *J. Dieudonné* (Evanston, Ill.).

Albert, A. A. On involutorial algebras. Proc. Nat. Acad. Sci. U.S.A. 41, 480–482 (1955).

Let A be an associative algebra with an involution $x\to x^*$ over a real field F, and suppose that there exists a linear functional $\delta(x)$ on A to F satisfying $\delta(xx^*)>0$ for every $x\neq 0$ in A. It is proved that (i) A is semisimple; (ii) for any $a=a^*$ in A, the characteristic roots of the right multiplication R_a are all real; (iii) if a in A is positive in the sense that $a=a^*\neq 0$ and $\delta(xax^*)\geq 0$ for every x in A, then the nonzero characteristic roots of R_a are all positive. The author's principal theorem is the following sharpened form of a conjecture of Weil: (iv) if F is the field of all rational numbers, then a in A is positive if and only if $a=x_1^2+x_2^2+x_3^2+x_4^2$ for $x_i=x_i^*$ in $F[a]$. *R. D. Schafer* (Storrs, Conn.).

Amitsur, S. A. Generic splitting fields of central simple algebras. Ann. of Math. (2) 62, 8–43 (1955).

A transcendental extension F of a field C is called regular over C if C is algebraically closed in F and F is separably generated over C. A mapping R of a field D into the union U of ∞ and a field E is called a place over C on D if $cR=c$ for c in C, $(a+b)R=aR+bR$, $(ab)R=(aR)(bR)$ for a and b in U, where we understand $a+\infty=\infty+a=\infty$ for every a of U, $a\infty=\infty a=\infty$ for every $a\neq 0$ of U. The image $(D)R$ is known as a residue field of D.

The author studies the set of all splitting fields (algebraic and transcendental) of a given central simple associative algebra A over a field C by the use of a theory of semilinear transformations. His main result may be stated as follows. Let A be central simple of degree n (order n^2) over C. Then there exists a field $D=D(A)\supseteq C$ such that D splits A; D is a regular extension of C of transcendence degree $n-1$; the splitting fields E of A are characterized by each of the following conditions: (a) a field $E\supseteq C$ splits A if and only if $E\supseteq E_0\supseteq C$ where E_0 is a residue field of D over C; (b) a field $E\supseteq C$ splits A if and only if the composite (E,D) is isomorphic to a field $E(x_1,\cdots,x_{n-1})$ of all rational functions in $n-1$ indeterminates x_i over E; the field D has a group of isomorphisms which is isomorphic to A^*/C^* where A^* and C^* are the groups of the regular elements of A and C respectively.

The author closes with a discussion of a geometric interpretation of his results. *A. A. Albert* (Chicago, Ill.).

Skornyakov, L. A. Representation of nonassociative rings in associative ones. Dokl. Akad. Nauk SSSR (N.S.) **102**, 33–35 (1955). (Russian)

Let A be an associative ring with operators Ω. Let A_1 be the ring obtained by adjoining a unit. Suppose we have a direct-sum decomposition of A_1 into Ω-subgroups: $A_1=B+C+\Omega_1$. We can define a multiplication in B by taking $b\circ b'$ to be the B-component of bb'. In this way a non-associative ring is obtained, and, reversing the point of view, we may speak of a representation of a non-associative ring in an associative one. The author presents five theorems concerning this type of representation; their statements are too lengthy for reproduction in a review. Proofs are included.

I. Kaplansky (Chicago, Ill.).

Stöcker, Claus. Beweis eines Hilfssatzes von Bruck und Kleinfeld unter Voraussetzung beliebiger Charakteristik. Arch. Math. **6**, 296–302 (1955).

In a paper of the reviewer and Kleinfeld on alternative rings [Proc. Amer. Math. Soc. **2**, 878–890 (1951); MR **13**, 526] a certain Lemma 4.1(i) was proved for characteristic not 2 and asserted (the proof being suppressed) to hold as well for characteristic 2. The unproved case was used implicitly by Kleinfeld [Proc. Nat. Acad. Sci. U.S.A. **37**, 818–820 (1951); MR **13**, 527]. The author fills the logical gap with a proof independent of characteristic. [Reviewer's remark. The proof is similar to the one suppressed, but longer. A simpler proof has since been given by M. Hall (see MR **17**, 73).]

R. H. Bruck (Madison, Wis.).

Schafer, R. D. Noncommutative Jordan algebras of characteristic 0. Proc. Amer. Math. Soc. **6**, 472–475 (1955).

An algebra A over a field F is called a (noncommutative) Jordan algebra if $(x^2y)x=x^2(yx)$, $(xy)x=x(yx)$ for every x and y of A. The author uses the reviewer's theory of traces to give a structure theory for all Jordan algebras of characteristic zero. He shows that all such algebras are trace-admissible, obtains the usual trace criterion for the radical (maximal nilideal) as well as the consequent decomposition of a semisimple algebra as a direct sum of simple algebras, and shows that the simple algebras are the simple commutative Jordan algebras, the simple flexible algebras of degree two, and the simple quasi-associative algebras. *A. A. Albert* (Chicago, Ill.).

Rees, D. A note on valuations associated with a local domain. Proc. Cambridge Philos. Soc. **51**, 252–253 (1955).

Soient Q un anneau d'intégrité local d'égales caractéristiques, et $\mathfrak{m}$ son idéal maximal. Une valuation v du corps des fractions F de Q est dite associée à Q si elle est discrète et de rang 1, et si elle admet $\mathfrak{m}$ pour centre sur Q [cf. Rees, "Valuations associated with local rings, I", à paraître aux Proc. London Math. Soc.]. L'intersection des anneaux des valuations associées à Q est la clôture intégrale de Q. Ceci se déduit aisément du résultat suivant: si $\mathfrak{a}$ est un idéal de Q, sa clôture intégrale est l'intersection des idéaux $\mathfrak{m}$-primaires et intégralement clos contenant $\mathfrak{a}$. *P. Samuel* (Clermont-Ferrand).

See also Lavendhomme, p. 61; Hochschild, p. 61; Hall, p. 73; André, p. 73; Boccioni, p. 74.

Groups, Generalized Groups

Munn, W. D., and Penrose, R. A note on inverse semigroups. Proc. Cambridge Philos. Soc. **51**, 396–399 (1955).

A semigroup S is called a generalized group by V. V. Vagner [Dokl. Akad. Nauk SSSR (N.S.) **84**, 1119–1122 (1952); MR **14**, 12] or an inverse semigroup by G. B. Preston [J. London Math. Soc. **29**, 396–403 (1954); MR **16**, 215] if it is regular ($axa=a$ solvable) and if any two idempotent elements of S commute with each other. Several alternative definitions are given in the present paper. The result that a semigroup S is an inverse semigroup if each element a of S has a unique inverse a' in S ($aa'a=a$ and $a'aa'=a'$) has also been given by A. E. Liber [Dokl. Akad. Nauk SSSR (N.S.) **97**, 25–28 (1954); MR **16**, 9]. *A. H. Clifford* (New Orleans, La.).

Thierrin, Gabriel. Demi-groupes inversés et rectangulaires. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 83–92 (1955).

Let D be a semigroup (French: "demi-groupe"). D is said to be inversed ("inversé") if to each a in D there exists x in D such that ax is idempotent; then there exists y in D such that both ay and ya are idempotent, namely $y=xax$. D is said to be rectangular if $ax=bx=ax'=m$ implies $bx'=m$. D is said to be stationary on the right if $x_1a=x_1b$ for some x_1 in D implies $xa=xb$ for all x in D. If D is rectangular, the set E of idempotent elements of D is a subsemigroup of D; $ee'=e'e$ (e, e' in E) implies $e=e'$; and $xey=xy$ for all x, y in D and e in E. A semigroup D stationary on either side is evidently rectangular; the converse holds if D is also inversed. If D is an inversed rectangular semigroup, then D^2 is simple and isomorphic with the direct product $G\times E$ of a group G and a rectangular semigroup E of idempotents; moreover, $aD\cong bD$ for arbitrary a and b in D. A semigroup is inversed and rectangular if and only if all its right regular equivalence relations are left stationary and vice-versa.

A. H. Clifford (New Orleans, La.).

Schiek, Helmut. Gruppen mit Relationen $(abc)^2=e$. Math. Nachr. **13**, 247–256 (1955).

Let the group $\mathfrak{G}$ be generated by the elements of a set $\mathfrak{S}$ subject to the defining relations $(abc)^2=1$ for all a, b, c, in $\mathfrak{S}$. Then the set $\mathfrak{T}$ of squares of elements in $\mathfrak{S}$ generates a normal subgroup $\mathfrak{Q}$ of $\mathfrak{G}$. The factor group $\mathfrak{G}/\mathfrak{Q}$ is a split extension of a free abelian group by a cyclic group of order 2, the generator of which induces the automorphism $x\to x^{-1}$. Defining relations of $\mathfrak{Q}$ are $X^3=(XY)^3=(XYXZ)^3=1$ for all X, Y, Z in $\mathfrak{T}$.

Graham Higman (Oxford).

Dwinger, Ph. On certain permutations of an abelian group. Simon Stevin **30**, 140–143 (1955).

Let G be an abelian group, G^2 be the subgroup of the squares of the elements of G, σ be an automorphism of G of period 2 generating an automorphism subgroup Σ, and form the relative holomorphs $H(G;\Sigma)$, $H(G^2;\Sigma)$ of Σ over G and G^2 respectively. It is immediate that the operation of replacing each element of G by its square induces a homomorphism of the first of these homomorphs onto the second with kernel isomorphic to G^2. If σ is the mapping of G which carries each element of G onto its inverse, then this holomorph over G^2 is isomorphic to the inner automorphism group of the holomorph of Σ over G.

F. Haimo (St. Louis, Mo.).

Yacoub, K. R. General products of two finite cyclic groups. Proc. Glasgow Math. Assoc. **2**, 116–123 (1955).

On the point set of the additive group of integers, modulo n, the author considers semispecial permutations π with the defining properties $\pi(0)=0$ and $\pi_u(x)=\pi(x+u)-\pi(u)=\pi^j(x)$ for every $x \in I_n$, the cyclic group of order n, where j is a function of u. Among his results are: if π^2 is the identity or if $\pi_u=\pi$ for some generator u of I_n, then there exists a generator r of I_n such that $\pi(x)=rx$. If $n>2$, he shows that there exists a divisor r of n with $1<r<n$ such that $\pi_r=\pi$. Semispecial permutations arise naturally in the study of group products of two finite cyclic groups A and B where $G=AB$ with $A \cap B$ trivial. Reference is made to J. Douglas [Proc. Nat. Acad. Sci. U.S.A. **37**, 604–610 (1951); MR **13**, 621]. *F. Haimo.*

Griffiths, H. B. A note on commutators in free products. II. Proc. Cambridge Philos. Soc. **51**, 245–251 (1955).

This note gives an improvement of a result in an earlier paper by the author [same Proc. **50**, 178–188 (1954); MR **15**, 775]. Let H be the ordinary free product of groups $G_1, \cdots, G_m$ and let $h = g_1 g_2 \cdots g_m$ with $g_i \in G_i$, $g_i \neq 1$. Then h is not the product of fewer than n commutators according to the first paper if $m \geq 12n-2$. The improvement here is to show that the conclusion holds if $m \geq n$. The proof is fairly complicated and uses various counting functions. *Marshall Hall, Jr.* (Columbus, Ohio).

Pic, G. Sur une nouvelle généralisation de la notion de nilpotence d'un groupe. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 199–211 (1954). (Romanian. Russian and French summaries)

As far as the reviewer has been able to make out, the author wishes to extend the concept of a nilpotent group in the following way: Let G be a group and Z_1 its maximal nilpotent normal subgroup (see remark (i) below). Define by recurrence Z_{i+1}/Z_i as the maximal nilpotent normal subgroup of G/Z_i. It can happen that $Z_k=G$ for a finite k (see remark (iii) below). Dually, take the intersection R_1 of all normal subgroups of G whose factor groups are nilpotent and define R_{i+1} as the intersection of all normal subgroups N of G for which R_i/N is the maximal nilpotent normal subgroup of G/N. It can happen that $R_k=1$ for a finite k (see remark (ii) below). In this case the author calls the group hypernilpotent and he studies some properties of hypernilpotent groups.

If this interpretation is correct, then the following remarks are in order: (i) Although the union of two nilpotent normal subgroups of an arbitrary group is itself nilpotent [Fitting, Jber. Deutsch. Math. Verein. **48**, 77–141 (1938) for finite groups; K. A. Hirsch, Proc. London Math. Soc. (2) **49**, 184–194 (1946) in the general case; MR **8**, 132], the union of all nilpotent normal subgroups of a group need not itself be nilpotent (Example: the restricted direct product of an infinite number of nilpotent groups of increasing classes). The maximal nilpotent normal subgroup exists, of course, if the maximal condition for normal subgroups is assumed to be satisfied. (ii) Similarly the intersection of all normal subgroups of a group whose factor group is nilpotent need not itself have a nilpotent factor group. (Example: in the free product of two cyclic groups of order 2 the intersection of all normal subgroups with a nilpotent factor group is the 1-element.) The minimal normal subgroup with nilpotent factor group exists, of course, if the minimal condition for normal subgroups is assumed to be satisfied. (iii) Under the additional assumptions the concept of hypernilpotency is precisely the same as that of solvability (K. A. Hirsch, loc. cit.) The proof of the author's principal theorem 11, p. 206 is the same as the one in the paper cited above, theorem 3.1. *K. A. Hirsch* (London).

Faddeev, D. K. On homology theory for finite groups of operators. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 193–200 (1955). (Russian)

Let G be a group, H a subgroup, A a G-module. For any integer $n \geq 0$ the elements $f \in C^n(G, A)$ which, in their homogeneous form, depend only on the right cosets of G mod H in which the arguments lie, form a subgroup $C_b^n(G, H, A)$ of $C^n(G, A)$; by definition $Z_b^n(G, H, A)=Z^n(G, A) \cap C_b^n(G, H, A)$, $B_b^n(G, H, A)=\delta C_b^{n-1}(G, H, A)$ and $H_b^n(G, H, A)=Z_b^n(G, H, A)/B_b^n(G, H, A)$. If H is normal in G then $H_b^n(G, H, A) \approx H^n(G/H, A^H)$. [These definitions and remark, but not the notation, are much as in Adamson, Proc. Glasgow Math. Assoc. **2**, 66–76 (1954) [MR **16**, 442] where with the added hypothesis that H is of finite index in G the definitions are extended to permit negative n.] Now let G have order hk, H have order h, h and k being relatively prime; because $hkH^n(G, A)=0$, there is a unique direct decomposition $H^n(G, A)=H_e^n(G, A)+H_i^n(G, A)$, where $kH_e^n(G, A)=0$ and $hH_i^n(G, A)=0$. The main result (Theorem 1) asserts that λ: $H_b^n(G, H, A) \approx H_e^n(G, A)$ and ι: $H_i^n(G, A) \approx H_e^n(H, A)$, where λ and ι are lift and restriction homomorphisms, and $H_e^n(H, A)$ is a certain direct summand of $H^n(H, A)$. As a corollary one has (Theorem 2), for any finite group G, $H^n(G, A) \approx \sum H_e^n(S_p, A)$ (direct sum), where p runs over the set of primes dividing the order of G, and S_p is a p-Sylow subgroup of G. The special case of Theorem 1 in which H is normal in G was given by Hochschild and Serre [Trans. Amer. Math. Soc. **74**, 110–134 (1953); MR **14**, 619]. *E. R. Kolchin* (New York, N.Y.).

Araki, Shôrô. On the homology of spinor groups. Mem. Fac. Sci. Kyûsyû Univ. Ser. A. **9**, 1–35 (1955).

The author uses a covering complex of the Whitehead-Miller cell complex in $SO(n)$ to investigate the homology of Spin (n). The boundary formula for this complex is calculated, and the Pontrjagin product which is compatible with the cellular structure is given explicitly. These two results provide the basis for all further calculations. Among the obtained results we mention: $H_{n+q}(\text{Spin } (n+1), \text{Spin } (n), G) \cong H_q(\text{Spin } (n), G)$ for all $q \geq 0$ and any coefficient group G; $H_i(\text{Spin } (n+1), \text{Spin } (n), G)=0$ for all $i<n$. [These are special cases of results of Wang, Duke Math J. **16**, 33–38 (1949); MR **10**, 468.] Further it is shown that the Pontrjagin ring $H_*(\text{Spin } (n), J_2)$ has a simple system of generators, the description of which is entirely analogous to a similar system for the cohomology ring $H^*(\text{Spin } (n), J_2)$ given by Borel [Amer. J. Math. **76**, 273–342 (1954); MR **16**, 219]. *W. T. van Est.*

Sekino, Kaoru. Über die Zerlegung der Gruppencharaktere. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 255–263 (1954).

In a former paper [Jap. J. Math. **18**, 729–744 (1943); cf. also ibid. **16**, 63–69 (1939); MR **7**, 372; **1**, 258] Z. Suetuna considered the character theory of a group G of which a factor group G/H is the linear group modulo p^n, for p an odd prime. Here the author examines the case $p=2$, $n \geq 3$, using, in particular, the transformations $H\tau=(z, z+1)$, $H\varrho=(z, 5^{-1}z)$, $H\pi=(z, -z)$ and their properties. *G. de B. Robinson* (Toronto, Ont.).

Murnaghan, Francis D. On the generation of the irreducible representations of the symmetric group. Proc. Nat. Acad. Sci. U.S.A. **41**, 514–515 (1955).

The author remarks, without proof, that the rth compound matrix of the matrix representation of the symmetric group on m symbols corresponding to the partition $(m-1, 1)$ of m is the representation corresponding to the partition $(m-r, 1^r)$. He deduces that all the irreducible representations of this group are expressible in terms of the invariant matrices of the representation corresponding to $(m-1, 1)$. Examples illustrate.

D. E. Littlewood (Bangor).

Murnaghan, Francis D. On the analysis of the Kronecker product of irreducible representations of S_n. Proc. Nat. Acad. Sci. U.S.A. **41**, 515–518 (1955).

The author proposes a new method of determining the analysis of the Kronecker product of two representations of a symmetric group, S_m. If $(\lambda)\equiv(\lambda_1, \cdots, \lambda_k)$ is a partition of p, the representation corresponding to $(m-p, \lambda_1, \cdots, \lambda_k)$ is denoted by (λ), and is said to be of depth p. If $(\lambda_1', \cdots, \lambda_{k'}')$ is a partition of $p'<p$, the expression $\{\lambda/\lambda'\}$ is used to denote the sum of the determinants obtained by decreasing the suffixes in k' rows of the determinant $\{\lambda\}=|h_{a-s+t}|$ by $\lambda_1', \cdots, \lambda_{k'}'$ respectively, in any order.

The author states that, in the reduction of the Kronecker product $(\lambda)\times(\mu)$, where (λ) is of depth p and (μ) of depth q, the terms of depth $p+q$ correspond to $(\{\lambda\}\{\mu\})$; the terms of depth $p+q-1$ to $\{\lambda/1\}\{\mu/1\}\{1\})$ the terms of depth $p+q-2$ to

$$(\{\lambda/1\}\{\mu/1\}+[\{\lambda/2\}\{\mu/2\}+\{\lambda/1^2\}\{\mu/1^2\}]\{2\} +[\{\lambda/2\}\{\mu/1^2\}+\{\lambda/1^2\}\{\mu/2\}]\{1^2\}).$$

An expression is given for the terms of depth $p+q-3$. The terms of depth $p-q$ are stated to be $\{\lambda/\mu\}$.

There is no proof, and the general form for the terms of depth $p+q-r$ is not obvious. Examples are given in which the terms of greater depth are calculated by the formulae while terms of smaller depth are inferred from specific examples. The method would appear to be useful, but needs clarification. *D. E. Littlewood* (Bangor).

Robinson, G. de B. On the modular representation of the symmetric group. V. Canad. J. Math. **7**, 391–400 (1955).

A class of the symmetric group on n symbols is said to be p-regular if the order of each element of the class is prime to p (not, as the author states, if the order of the class is prime to p). It is known that the number of p-regular classes is equal to the number of partitions (λ) of n in which no summand is repeated p or more times. The author proves that in any p-block the number of p-regular partitions is equal to the number of modular irreducible representations included in the block.

This result is proved by the study of the r-Boolean algebra of a partition (λ) [G. de B. Robinson, same J. **6**, 486–497 (1954); MR **16**, 333], which is obtained by adding and deleting r-nodes in the diagram of the partition. The complement of $[\lambda]$ is obtained by removing the d^* r-nodes and adding d r-nodes in the d free r-positions. If the complement of a nonsingular partition is singular the modified complement is obtained by transferring the r-node which makes the partition singular to the other end of a kr-hook. The property that a diagram be regular is then invariant under modified complementation.

An explicit construction is given of a Young diagram with given p-core and p-quotient. It is shown that the number of p-regular diagrams of weight w with a given p-core is equal to

$$\sum_{w_1\cdots w_{p-1}} p_{w_1}p_{w_2}\cdots p_{w_{p-1}}, \quad \Sigma w_i=w, \quad 0\le w_i\le w,$$

where p_i is the number of partitions of i, and is thus equal to the number of modular irreducible representations of S_n in the given p-block. *D. E. Littlewood* (Bangor).

Artin, Emil. The orders of the linear groups. Comm. Pure Appl. Math. **8**, 355–365 (1955).

Soit $N(q, n)$ l'ordre du groupe projectif unimodulaire correspondant à un espace vectoriel de dimension n sur un corps à q éléments (groupe simple sauf pour $n=2$, $q=2$ ou 3). Il y a trois cas connus où $N(q, n)=N(q_1, n_1)$ pour des couples $(q, n)\neq(q_1, n_1)$ et six où $N(q, n)=m!/2$ (ordre du groupe alterné sur m lettres). L'auteur prouve que ces cas sont les seuls. Pour la seconde relation il suffit de remarquer que $q^{n(n-1)/2}$ est la plus grande puissance de p ($q=p^r$) divisant N et que l'on a $N<q^{n^2-1}$; on connaît d'autre part la plus grande puissance de p divisant $m!$, et de ces remarques il est facile de déduire l'inégalité $m<e\cdot p^{p/(p-1)}$, d'où on conclut aisément que m est borné, et il n'y a donc qu'un nombre fini de cas à examiner (que l'auteur réduit d'ailleurs beaucoup par diverses remarques ingénieuses). En ce qui concerne la relation $N(q, n)=N(q_1, n_1)$, le principe de la méthode est le même, mais il faut examiner un plus grand nombre de possibilités, et s'appuyer essentiellement sur les propriétés de divisibilité des nombres q^k-1, que l'auteur commence par rappeler en un paragraphe préliminaire. On élimine aisément le cas où $p_1=p$ ($q_1=p_1^{r_1}$); si f est l'exposant de q mod p_1 (en supposant par exemple p_1 impair et $p_1\neq p$), on montre ainsi que la plus grande puissance de p_1 qui divise $(q^n-1)(q^{n-1}-1)\cdots(q-1)$ est au plus $q^n\cdot 3^{n/2}$; cela permet de montrer que l'on ne peut avoir à la fois $n\ge 4$ et $n_1\ge 4$. Il reste enfin à examiner séparément les cas $n=2$ et $n=3$, ce qui se fait par des méthodes semblables. *J. Dieudonné* (Evanston, Ill.).

Loonstra, F. L'extension du groupe ordonné des entiers rationnels par le même groupe. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17**, 41–49 (1955).

A group G is called partially ordered if for any two elements a and b of the group one can determine if the elements a and b are ordered or not in such a way that: 1) $a\ge a$ for any a; 2) $a\ge b$ and $b\ge c$ implies $a\ge c$; 3) $a\ge b$ implies $x+a+y\ge x+b+y$ for any x and y belonging to G. An invariant subgroup N of G is called convex if the inequality $a\ge x\ge b$ implies $x\in N$ when $a\in N$, $b\in N$ and $a\ge b$. The author considers the following problem. Let C and F be ordered groups of rational numbers. Does there then exist an extension G (in Schreier's sense) of C by F such that C is a convex invariant subgroup of G and G is partially ordered. Following ideas of L. Fuchs, the author defines partial order for an extension. In the case $C=F$ he determines all possible partially ordered extensions of C by F. He also defines the concept of equivalence of two extensions and gives necessary and sufficient conditions for the equivalence in the case $C=F$.

H. Bergström (Göteborg).

See also: Ganea, p. 60; Obreanu, p. 60; Harish-Chandra, p. 60; Naĭmark, p. 61; van Est, p. 61; Bartle, p. 62; Hall, p. 73.

THEORY OF NUMBERS

★ **Vinogradov, I. M. An introduction to the theory of numbers.** Translated by H. Popova. Pergamon Press, London & New York, 1955. vi+155 pp. $1.75; 12s. 6d.

Translation of the author's Osnovy teorii čisel, 6th ed. [Gostehizdat, Moscow, 1952; MR **15**, 601].

Nagell, Trygve. Contributions to the theory of a category of Diophantine equations of the second degree with two unknowns. Nova Acta Soc. Sci. Upsal. (4) **16**, no. 2, 38 pp. (1955).

Throughout the discussion D will denote a positive integer not a perfect square, and C a square-free integer. The first part of the paper treats the elementary properties of the Diophantine equations $u^2-Dv^2=\pm 4$ without the use of the theory of quadratic fields or binary quadratic forms. This development parallels the author's treatment of $x^2-Dy^2=\pm 1$ as given in his book [Introduction to number theory, Wiley, New York, 1951; MR **13**, 207]. Next it is established that if (1) $u^2-Dv^2=C$ has an ambiguous class of solutions (definitions may be found in the book cited above, pp. 204–208) then C divides $2D$. On the other hand, if C divides $2D$, then equation (1) has at most one class of solutions, and this class, if it exists, is ambiguous.

A parallel result is obtained for the equation (2) $u^2-Dv^2=4C$. A basic theory for (2) along lines similar to Nagell's work on (1) has been given by B. Stolt [Ark Mat. **2**, 1–23 (1952); MR **14**, 247]. Throughout the present paper much use is made of the parallel between equations (1) and (2), so that almost every theorem has an analogue; in several instances one member of such a pair of analogous theorems is an already known result.

Say that C is admissible if it is a divisor of $2D$, with $C\neq 1$ and $C\neq -D$. If C is admissible then any solution u, v of (1) gives a solution $x=(u^2+Dv^2)/|C|$, $y=2uv/|C|$ of $x^2-Dy^2=1$; moreover, if the solution u, v is least positive among solutions of (1), then the corresponding x, y is also least positive. Furthermore, for a given D which is not square-free, there is at most one admissible value of C for which (1) is solvable. For a given D which is square-free there are exactly two admissible values of C for which (1) is solvable; the product of these two values of C is $-4D$ when D is odd and C is even, and the product is $-D$ in all other cases.

Next we assume that D is odd and that C divides D. If a solution u, v in positive odd integers of (2) has the property that all prime factors of v are factors of D, then it is the least positive solution in odd integers of (2); exception must be made in the case $D=5$, $C=-1$, which is fully treated. This is an analogue of a result of K. Mahler [Arch. Math. Naturvid. **41**, no. 6 (1935)]. These results are generalized (we omit the details) and also applied to obtain an analogue of another result of Mahler, as follows. A method is given of finding all solutions, there being only a finite number, in positive integers $x, z_1, z_2, \cdots, z_m$ of the equation $Ax^2-4E=q_1^{z_1}q_2^{z_2}\cdots q_m^{z_m}$, where $E=\pm 1$, A is positive, odd, and square-free, and $q_1, q_2, \cdots, q_m$ are any given distinct odd primes which do not divide A.

Application is also made to obtain the following result. We are given n odd, $n\geq 3$, A odd, square-free, $A\geq 3$; assume that n is not a divisor of the class number of the field $R(\sqrt{-2A})$. Then $Ax^2+2=y^n$ has no solutions in integers. Related equations were treated earlier by the writer and by W. Ljunggren [Norske Vid. Selsk. Forh., Trondhjem **15**, no. 30, 115–118 (1942); MR **8**, 442]. Another analogue of a result of Mahler deals with any polynomial $f(x)=Ax^2\pm 4$, where A is a positive integer. For any integer x define $P(x)$ as the greatest prime factor of $f(x)$. Then $\liminf P(x)/\log\log x\geq 1$.

The paper is partly expository, with new proofs of various known results, and provides a fairly unified treatment of the subject. *I. Niven* (Eugene, Ore.).

Klöter, Hubert, und Stöhr, Alfred. Wesentliche Komponenten und asymptotische Dichte. J. Reine Angew. Math. **194**, 210–217 (1955).

For any non-negative set of integers B, write $B(n)$ for the number of elements of B which do not exceed n, so that the asymptotic density of B is $\inf B(n)/n$. Define $k(n)$ as the smallest integer k such that n is a sum of k elements of B. Assume that $k(n)$ exists for all sufficiently large n, say $n\geq n_0$, and define $h=\sup k(n)$ and $\lambda=\sup n^{-1}\sum_{j=n_0}^{n} k(j)$. Assume that λ is finite. Then $h\leq 2\lambda$, and if $0\in B$ we have

$$\lambda=h-\inf\frac{1}{n}\sum_{q=1}^{h-1}B_q(n),$$

where $B_q(n)$ is the number of elements of the set $B+B+\cdots+B=qB$ which do not exceed n. Let α and γ denote the asymptotic densities of the sets A and $A+B$. If $0\in B$ then $\gamma\geq\alpha\{1+(1-\sqrt{\alpha})/\lambda\}$. Several corollaries of these propositions are also obtained.

I. Niven (Eugene, Ore.).

Rohrbach, Hans, und Volkmann, Bodo. Verallgemeinerte asymptotische Dichten. J. Reine Angew. Math. **194**, 195–209 (1955).

Let $\varphi(x)$ be a monotone increasing function satisfying the conditions $\varphi(x)>0$ for $x\geq 0$, $\varphi(x)=0$ for $x<0$. We put $\Phi(x)=\sum_{j=1}^{x}\varphi(j)$ and for every set $\mathfrak{A}$ of integers

$$\varphi A(x)=\sum_{a\leq x,\, a\in\mathfrak{A}}\varphi(a),\quad D_\varphi(\mathfrak{A})=\liminf_{x\to\infty}\varphi A(x)/\varphi(x).$$

Let φ satisfy the following three conditions.

(1) $\lim_{x\to\infty}\varphi(x+c)/\varphi(x)=1$ for every c.

(2) $D_\varphi(\mathfrak{A}h)=1/hD_\varphi(\mathfrak{A})$ for every set $\mathfrak{A}$, where $\mathfrak{A}h=\{a_1h, a_2h, \cdots\}$.

(3) $\varphi(x+1)^2\geq\varphi(x)\varphi(x+2)$.

The author generalizes the theorem of Kneser on the asymptotic density [Math. Z. **58**, 459–484] (1953); MR **15**, 104] and shows that the asymptotic density in Kneser's theorem can be replaced by D_φ. The methods of Kneser carry through with some additional considerations. The author also shows that for rational and pseudorational sets [for definitions see Volkmann, J. Reine Angew. Math. **190**, 199–230 (1952); MR **15**, 15] $D_\varphi(\mathfrak{A})=\delta(\mathfrak{A})$, where $\delta(\mathfrak{A})$ is the asymptotic density. On the other hand, he gives an example for which $D_\varphi(\mathfrak{A})\neq\delta(\mathfrak{A})$.

There are two points of error about the history of the problem. The e transformation was first introduced by Dyson as a modification of the reviewer's method and not by van der Corput, who only gave it its name. There is also an ambiguous statement which might give rise to the erroneous belief that some of the ideas in the reviewer's proof of the fundamental theorem were suggested to him by another worker in the field.

H. B. Mann (Columbus, Ohio).

Venkata Narayana, Tadepalli. Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités. C. R. Acad. Sci. Paris **240**, 1188–1189 (1955).

Denote a solution of the equation (1) $t_1+t_2+\cdots+t_r=n$ in integers $t_i\geq 1$ by $n(t_1, t_2, \cdots, t_r)$. A solution $n(t_1, t_2, \cdots, t_r)$ is said to dominate the solution $n(t_1', t_2', \cdots, t_r')$ if and only if $t_1\geq t_1'$, $t_1+t_2\geq t_1'+t_2', \cdots, t_1+\cdots+t_{r-1}\geq t_1'+\cdots+t_{r-1}'$. If $p_1, p_2, \cdots, p_a$, $a=C_n^{r-1}$, denote the distinct solutions of (1) in some order and x_i the numbers of solutions dominated by p_i, then the relation $x_1+\cdots+x_a=C_n^r C_n^{r-1}/n$ is stated. It is further shown how this relation can be used when considering a special game of coin-tossing. *H. Bergström* (Göteborg).

Prachar, K. Über die Lösungszahl eines Systems von Gleichungen in Primzahlen. Monatsh. Math. **59**, 98–103 (1955).

In a previous paper [J. London Math. Soc. **29**, 347–350 (1954); MR **15**, 935] the author proved some inequalities for the number of representations of an even integer as the sum of two primes. He now considers $p_1+p_2=n$, $p_3+p_3=m$, $m\neq n$, where p_1, p_2, p_3 are primes, and obtains similar results for the number of solutions of this system of equations. *R. D. James* (Vancouver, B.C.).

Hornfeck, Bernard. Berichtigung zur Arbeit: Ein Satz über die Primzahlmenge. Math. Z. **62**, 502 (1955).

In the paper referred to [Math. Z. **60** (1954), 271–273; MR **16**, 221] the author used an estimate due to Knödel [Monatsh. Math. **55** (1951), 62–75; MR **12**, 676; **14**, 1277] to prove that the set of prime numbers is totally irreducible. This estimate contains a coefficient c depending on a parameter k, which the author replaced by a constant in order to obtain his result. His arguments justifying this replacement are valid when applied to the main term $\prod L_i$ of Knödel's estimate, but do not apply to the error term. The misunderstanding presumably arises from the fact that, although Knödel's result (Satz 2, top of p. 63) is correct as stated, the coefficient c_{18} at the bottom of p. 70 should be replaced by $c_{18}(k)$ in order to incorporate the estimate for the error term derived in § 3. The present note points out the error. *R. A. Rankin* (Glasgow).

Erdős, Pál. On a problem concerning congruence systems. Mat. Lapok **3**, 122–128 (1952). (Hungarian. Russian and English summaries)

The set of congruences $x\equiv a_i \pmod{n_i}$, $1<n_1<n_2<\cdots<n_k$, is called overlapping if every positive integer satisfies at least one of them. Using this concept, the author proves the existence of an arithmetic progression consisting of odd integers none of which is representable as the sum of a prime and a power of two.

P. R. Halmos (Chicago, Ill.).

Harris, V. C. An analog of an identity of Jacobi. Bol. Mat. **28**, 17 (1955).

By subtracting an identity of Jacobi from an identity of Euler the author obtains the identity

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\{5-(-1)^n\}nx^n/(1-x^n)=\sum\sigma(2k)x^k,$$

where $\sigma(m)$ is the sum of the divisors of m. The identity is equivalent to the statement

$$\sigma(2k)=2\sigma(k)+\sigma_0(k),$$

where the last term is the sum of the odd divisors of k.

D. H. Lehmer (Berkeley, Calif.).

Cassels, J. W. S., and Swinnerton-Dyer, H. P. F. On the product of three homogeneous linear forms and indefinite ternary quadratic forms. Philos. Trans. Roy. Soc. London. Ser. A. **248**, 73–96 (1955).

The authors prove "local isolation" theorems in which the concept of "neighbourhood of a form f" is used. If $(x_1, \cdots, x_n)$ is an algebraic form of some specific type, then for any $\varepsilon>0$ an ε-neighbourhood of f is defined as the set of all forms f^* of the same type whose coefficients lie within the usually defined ε-neighbourhood of the corresponding coefficients of f. Any set which contains some ε-neighbourhood of f is called a neighbourhood. Main result: Let $f(x, y, z)=L_1L_2L_3$ be the product of three homogeneous linear forms in x, y, z which represent zero only for $x=y=z=0$ and suppose that f has integer coefficients. Let (δ_1, δ_2) be any open interval. Then there is a neighbourhood of f, such that all forms in that neighbourhood which are not multiples of f itself, take some value in (δ_1, δ_2) for integer values of x, y, z. This theorem holds for neighbourhoods in the set of all ternary cubic forms; the authors however restrict themselves to the proof of the case of neighbourhoods in the set of all products of three linear forms. Further: Let L_1, L_2, L_3 be the forms of the above theorem and let L_2^*, L_3^* be any real linear forms such that the product $L_2^*L_3^*$ is not a numerical multiple of L_2L_3. Then the set of values taken by $L_1L_2^*L_3^*$ for integer values of x, y, z, is everywhere dense in $(-\infty, +\infty)$.

The authors further state several hypotheses on the product of three linear forms which at the moment neither can be proved nor denied, and deduce some theorems concerning the equivalence, etc. of these hypotheses. *J. F. Koksma* (Amsterdam).

Analytic Number Theory

Rodosskiĭ, K. A. The exceptional zero and the distribution of prime numbers in short arithmetic progressions. Mat. Sb. N.S. **36(78)**, 341–348 (1955). (Russian)

The following two theorems are proved. (1) Let $(D, l)=1$, $14\ln D\geq\ln x\geq\ln^3 D$. Then positive absolute constants D_1, A_1, A_2 exist such that

$$\left|\Psi(x, D, l)-\frac{x}{\varphi(D)}\{1-E(D)\tilde{\chi}(l)\tilde{\beta}x^{\tilde{\beta}-1}\}\right| < \frac{x}{\varphi(D)}\left\{A_2\ln^{12} D\left(\frac{lA_1}{\delta_0\ln D}\right)^{-A_1\ln\frac{1}{2}x\ln D}+\frac{1}{D}\right\}$$

for $D>D_1$. $\varphi(D)$ is Euler's φ-function, and

$$E(D)=1,\ \delta_0=1-\tilde{\beta} \quad \text{if } 1-\tilde{\beta}\geq A_1\ln^{-1} D,$$
$$E(D)=0,\ \delta_0=A_1\ln^{-1} D \text{ if } 1-\tilde{\beta}>A_1\ln^{-1} D.$$

(2) Positive absolute constants D_2, A_3, A_4, A_5 exist such that if $x^{1-A_5}\leq u\leq x$, $\ln x/\ln\ln x\geq A_4\ln D$ and $D\geq D_2$, then

$$\left|\Psi(x+u, D, l)-\Psi(x, D, l)-\frac{u}{\varphi(D)}\{1-E(D)\tilde{\chi}(l)x^{\tilde{\beta}-1}\}\right| < A_3\delta_0\frac{u}{\varphi(D)},$$

$E(D)$ and δ_0 being defined as above. The proofs of the above make use of an improved estimate for $N(\Delta, T, D)$ for $0.9\leq\Delta\leq 1$, $T\geq 3$, $\ln D\geq 12$. *W. H. Simons.*

Beurling, Arne. A closure problem related to the Riemann zeta-function. Proc. Nat. Acad. Sci. U.S.A. **41**, 312–314 (1955).

Let $\varrho(x)$ be the fractional part of x and let C be the set of functions $f(x)=\sum_1^n c_\nu\varrho(\theta_\nu x^{-1})$, where $0<\theta_\nu\leq 1$, $\sum_1^n c_\nu\theta_\nu=0$. Let C_p be the closure of C in $L_p(0,1)$, $1<p<\infty$. Here $C_p=L_p$ if and only if the function $f(x)\equiv 1$, $0<x<1$, is in C_p. It is shown that the Riemann zeta-function has no zeros in $\Re(s)>1/p$ if and only if $C_p=L_p$. The sufficiency follows from the relation $s\int_0^1 x^{s-1}f(x)dx=-\zeta(s)\sum_1^n c_\nu\theta_\nu^s$ valid for any $f(x)\in C$. The necessity is concluded by observing that if $C_p\neq L_p$, then among the functions of L_q, $1/p+1/q=1$, which are orthogonal to C, there is also a power x^λ, $\Re(\lambda)>-1/q$ and if $s_0=\lambda+1$, then $\zeta(s_0)=0$ where $\Re(s_0)>1/p$. Here x^λ is a character of the semi-group of positive reals ≤ 1 under multiplication and the set C is mapped into itself by the operators $\{T_a;\ 0\leq a\leq 1\}$ which take $f(x)$ into $f(x/a)$ where $f(x)=0$ for $x>1$. *E. Hille* (New Haven, Conn.).

Kopřiva, Jiří. On a relation of the Farey series to the Riemann hypothesis on the zeros of the ξ function. Časopis Pěst. Mat. **78**, 49–55 (1953). (Czech)

It is proved that each of the estimates

$$\sum_{0<r<\frac{1}{2}}(r-\tfrac{1}{4})\ll q^{\frac{1}{2}+\varepsilon},\quad \sum_{0<r<1}(r^2-\tfrac{1}{3})\ll q^{1+\varepsilon},\quad \sum_{0<r<1}(r^3-\tfrac{1}{4})\ll q^{1+\varepsilon},$$

where $\varepsilon>0$ is arbitrarily small and the sums extend over all Farey fractions r lying in the indicated intervals with denominators $\leq q$, are equivalent to the relation

$$\sum_{n\leq q}\mu(n)\ll q^{\frac{1}{2}+\varepsilon}$$

(the sign $\ll$ of I. M. Vinogradov is taken, as usual, in the sense O). Here $\mu(n)$ is the Möbius function. By a known theorem of Littlewood [C. R. Acad. Sci. Paris **154**, 263–266 (1912)] follows the equivalence of these estimates to the Riemann hypothesis.

These results are analogous to known results of Franel [Landau, Vorlesungen über Zahlentheorie, Bd. 2, Hirzel, Leipzig, 1927, pp. 167–177].

I. P. Kubilyus (RŽMat **1955**, no. 595).

Kopřiva, Jiří. Contribution to the relation of the Farey series to the Riemann hypothesis. Časopis Pěst. Mat. **79**, 77–82 (1954). (Czech)

Let $M(x)$ denote the sum $\sum_{M<x}\mu(m)$, where μ is the Möbius function. The statement (1) $M(x)=O(x^{\frac{1}{2}+\varepsilon})$ is known to be equivalent to the Riemann Hypothesis. The author is interested in obtaining transformations of this statement (1). In the present paper he introduces the function $H(q)=\sum_{m<1}^{q}m^{-k}M(q/m)$, where k is an integer parameter and shows that (1) is equivalent to the statement $H(q)=O(q^{\frac{1}{2}+\varepsilon})$, and that this is equivalent to the statement

$$\sum_{\varrho}(\varrho^{n-1}-n^{-1})=O(q^{\frac{1}{2}+\varepsilon})\quad(n=2,3,\cdots),$$

where the sum extends over all Farey fractions whose denominators do not exceed q. This last statement for $n=2,3$ was given in the paper (unavailable to the reviewer) reviewed above. *D. H. Lemher.*

Newman, Morris. An identity for the coefficients of certain modular forms. J. London Math. Soc. **30**, 488–493 (1955).

The author considers the coefficients $p_r(n)$ in the development

$$\prod_{n=1}^{\infty}(1-x^n)^r=\sum_{n=0}^{\infty}p_r(n)x^n.$$

If r is one of the numbers 2, 4, 6, 8, 10, 14, 26 and p is a prime >3 such that $r(p+1)\equiv 0 \pmod{24}$, he proves that $p_r(np+\Delta)=(-p)^{(r/2)-1}p_r(n/p)$, where $\Delta=r(p^2-1)/24$ and $p_r(\alpha)$ is zero if α is not a nonnegative integer. There are no other values of r for which the formula is true.

H. D. Kloosterman (Ann Arbor, Mich.).

★ **Cohn, Harvey. Modular functions defined by perturbation mappings.** Lectures on functions of a complex variable, pp. 341–348. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Let Π_ε be the region in the upper half z-plane which remains after removing from it all interiors of the circles $C_{p/q}(\varepsilon)$ tangent to the real axis at p/q and of diameter ε/q^2. Here p/q runs through the rational points on the real axis; $0<\varepsilon<1$; $C_{1/0}(\varepsilon)$ is the region $\operatorname{Im} z>1/\varepsilon$. If $\zeta=\zeta_\varepsilon(z)$ is the function which maps Π_ε onto the upper half ζ-plane with point and directions fixed at $z=\zeta=i$ ($d\zeta/dz>0$), the author proves that the limit

$$\lim_{\varepsilon\to 0}\frac{\zeta_\varepsilon(z)-z}{\varepsilon^2}$$

exists and is equal to $-\pi^2P(z)/12$, where

$$P(z)=z^3-5z/2-\sum_{q=1}^{\infty}\sum_{p}q^{-3}(p+qz)^{-1}$$

(p runs through all integers prime to q). The third derivative of $P(z)$ is an Eisenstein series.

H. D. Kloosterman (Ann Arbor, Mich.).

Rademacher, Hans. On the transformation of $\log\eta(\tau)$. J. Indian Math. Soc. (N.S.) **19**, 25–30 (1955).

The author provides an elegant proof of the difficult transformation formula for the Dedekind modular function $\log\eta(\tau)$, where for $\operatorname{Im}\tau>0$,

$$\eta(\tau)=\exp\{\pi i\tau/12\}\prod_1^{\infty}(1-\exp\{2\pi i m\tau\}).$$

He states that his proof is a generalization of one of Siegel's [Mathematika **1**, 4 (1954); MR **16**, 16] where the special transformation $\tau\to-1/\tau$ was considered, but aside from the fact that contour integration and the calculus of residues are used in both, the two proofs have little in common.

The general transformation formula for

$$\log\eta(\tau)\quad(\tau\to(a\tau+b)/(c\tau+d))$$

can be written as an identity in which the right member is $-\log z$, while the left member involves Lambert series, an elementary term, and the arithmetic function studied by Dedekind: $s(h,k)=\sum_{\mu=1}^{k-1}(\mu k^{-1}-\frac{1}{2})(\mu hk^{-1}-[\mu hk^{-1}]-\frac{1}{2})$. Here $\tau=(h+iz)/k$, h/k runs over all reduced rational fractions in (0, 1) and $\operatorname{Re} z>0$. Rademacher introduces the function (of which the first term is essentially Siegel's)

$$F_n(x)=-\frac{1}{4ix}\coth\pi Nx\cot\frac{\pi Nx}{z}+\sum_{\mu=1}^{k-1}\frac{1}{x}\cdot\frac{e^{2\pi i\mu Nz/k}}{1-e^{2\pi iNz}}\cdot\frac{e^{2\pi i\mu^*Nz/kz}}{1-e^{2\pi iNz/z}}$$

with $N=n+\frac{1}{2}$, and the parallelogram C with vertices z, i, $-z$, $-i$ ($\mu^*\equiv h\mu\pmod k$, $1\leq\mu^*<k$). By direct estimation and with $n\to\infty$, we find $\lim_{n\to\infty}\int_C F_n(x)dx=-\log z$. The remainder of the proof consists in showing that the residues of $F_n(x)$ add up, when $n\to\infty$, to the left member of the above-mentioned identity, which is done by straight-

forward calculation. The residue at $x=0$ of the general term in the sum of $F_n(x)$ contains the term

$$(\mu k^{-1}-\tfrac{1}{2})(\mu^* k^{-1}-\tfrac{1}{2}).$$

Since from the definition we have $\mu^* k^{-1}=h\mu k^{-1}-[h\mu k^{-1}]$, this is the way the Dedekind sum $s(h, k)$ arises in the final formula.

This proof is shorter and more elementary than the previously known proofs of the Dedekind formula.

J. Lehner (Los Alamos, N.M.).

Guinand, A. P. Some rapidly convergent series for the Riemann ξ-function. Quart. J. Math. Oxford Ser. (2) **6**, 156–160 (1955).

The author derives the following identity, valid for $\mathrm{Re}\, z>0$ and for all s except 0 or ± 1,

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)K_{\frac{1}{2}s}(2\pi nz)-z^{-1}\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)K_{\frac{1}{2}s}(2\pi n/z)$$
$$=\frac{\xi(s)}{2s(s-1)}(z^{\frac{1}{2}s-1}-z^{-\frac{1}{2}s})+\frac{\xi(-s)}{2s(s+1)}(z^{-\frac{1}{2}s-1}-z^{\frac{1}{2}s}),$$

where $\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\xi(s)$, $\sigma_s(n)=\sum_{d|n} d^s$, and K_μ is a Bessel function. Repeated differentiation of this formula leads to the following formula, valid for $s\neq 0$:

$$\xi(s)=4\pi(1-s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1+\frac{1}{2}s}(2n\pi)+$$
$$16\pi^2 s^{-1}\sum_{n=1}^{\infty} n^{2-\frac{1}{2}s}\sigma_s(n)\{(s-7)K_{1+\frac{1}{2}s}(2n\pi)+(s+7)K_{1-\frac{1}{2}s}(2n\pi)\}.$$

The series on the right converges very rapidly and hence is particularly useful for computing $\xi(s)$.

The author also obtains rapidly convergent series for $\xi(2p-1)$ (p=integer>1) as a consequence of a transformation formula for the Lambert series

$$\sum_{n=1}^{\infty} n^{1-2p}(e^{2\pi ns}-1)^{-1}$$

which he derived in an earlier paper [same J. **15**, 11–23 (1944); MR **5**, 263]. The last term in formula (9) should contain z^{2p-2} instead of z^{2p-1}.

T. M. Apostol.

Algebraic Number Theory

Hasse, H. Über das Problem der Primzerlegung in Galoisschen Zahlkörpern. S.-B. Berlin. Math. Ges. **1951/52**, 8–27 (1952).

The history and present state of the problem of decomposition into primes in algebraic number fields are reviewed. The problem is introduced as the problem of a characterization, in the constructive sense, of the set $\mathfrak{M}(n_1, n_2, \cdots, n_r)$ of prime numbers modulo which a given monic irreducible polynomial f (of degree $n=n_1+n_2+\cdots+n_r$) with rational-integral coefficients is decomposed according to the "decomposition type" $(n_1, n_2, \cdots, n_r)$. The Frobenius density theorem is given, which asserts that the density of the set $\mathfrak{M}(n_1, n_2, \cdots, n_r)$ relative to the set of all primes not dividing the disdriminant of f is equal to the relative frequency in the Galois group of f, given as a permutation group of roots of f, of permutations of type $(n_1)\ (n_2)\cdots(n_r)$. On translating the problem into invariant terminologies of prime ideal decomposition in an algebraic number field, and introducing Frobenius' substitution, Tschebotarew's finer density theorem is stated. The cases hitherto solved, the quadratic case ($n=2$) of Euler and Gauss, Kummer's case of the field $\mathsf{P}(\sqrt[n]{1})$ (P being the rational number field), Kronecker's case of general cyclotomic fields (over P), are discussed in conjunction with the quadratic reciprocity law, Dirichlet's prime-number theorem and Takagi's general class-field theory. The metabelian case, with a solvable Galois group, has been investigated in many instances by the author and his followers and is accessible in principle by reduction to the relative abelian case; but to state the decomposition law, in the general metabelian case, in a form independent of the choice of composition series of the Galois group is an open problem. As a case for which such has been done, the case of cubic fields is noted, the solution being in terms of the quadratic-residue symbol and quadratic forms. Next the author's [J. Reine Angew. Math. **187**, 14–43 (1949); MR **11**, 576] theory of Galois algebras and its reformulation by Wolf [Abh. Math. Sem. Univ. Hamburg **18**, 179–195 (1952); MR **14**, 531] are alluded to, to the effect to generalize the theory of Kummer fields. Noether's theory of crossed products and the Brauer-Hasse-Noether principal theorem of algebras over algebraic number fields are introduced, and the characterization, not in the above sense however, of primes completely splitting in a Galois field K as primes at which every crossed product (K, γ) associated with K splits locally is given. Further, Dirichlet's L-functions, associated with characters of an abelian Galois group, are introduced, and together with the conductor-discriminant formula, the factorization of Dirichlet's ζ-function into the L-function is discussed as a reformulation of the decomposition law. Then Artin's L-functions and conductors, associated with characters of a general non-abelian Galois group, are introduced; the secret of the general decomposition law, if it exists, lies hidden in the open problems of finding the analogy for the former of the sum-expression of Dirichlet's L-functions and the significance of the latter as congruence modules. Finally the author's [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1952**, no. 1; MR **15**, 404] theory of generalized Gauss sums, associated with characters of the Galois group and defined as factors in the functional equations of L-functions, is cited. Such a generalized Gauss sum can be decomposed into a product of abelian ones, its absolute value is the square root of [the absolute norm of (in the relative case)] the conductor with the same character, and its formal properties are similar to those of the L-function and the conductor. Here the secret of the decomposition law in question lies in the problem of finding the analogy for such a generalized Gauss sum of the ordinary expression as a finite sum of roots of unity for an ordinary Gauss sum.

T. Nakayama (Zbl **49**, 27.)

Cohen, Eckford. The quadratic singular sum. Duke Math. J. **22**, 373–381 (1955).

Let F be any finite extension of the rational field, A an arbitrary odd ideal of F and α_i ($i=1, i, \cdots, s$) integers of F which are prime to A. Then the number of solutions of the congruence

$$\varrho\equiv\alpha_1\xi_1^2+\cdots+\alpha_s\xi_s^2 \pmod{A}$$

for any integer ϱ of F is equal to

$$\mathfrak{S}_s(\varrho, A, \alpha)=\begin{cases} N^{2m-1}(A)\sum_{D|A}\left(\frac{\alpha}{D}\right)\frac{R(\varrho, D)}{N^m(D)} & (s=2m),\\ N^{2m}(A)\sum_{D|A}\left(\frac{\alpha}{D}\right)\frac{J(\varrho, D)S(1, D)}{N^{m+1}(D)} & (s=2m+1),\end{cases}$$

where

$$\alpha=(-1)^m\alpha_1\alpha_2\cdots\alpha_s \quad \text{for } s=2m,$$
$$\alpha=(-1)^{m+1}\alpha_1\alpha_2\cdots\alpha_s \text{ for } s=2m+1,$$

and $R(S, D)$ is the Rademacher sum and $J(\varrho, D)$ a sum related to the Hecke sum. Further, $G_s(\varrho, A, \alpha)$ is given in an arithmetic form. These results generalize earlier results of the author. *H. Bergström* (Göteborg).

Springer, T. A. Quadratic forms over fields with a discrete valuation. I. Equivalence classes of definite forms. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indag. Math. **17**, 352–362 (1955).

The author considers quadratic forms over a commutative field K which is complete for a non-Archimedian discrete valuation, $|\lambda|$, and E a vector space over K with finite dimension n. Since it has been shown by Witt that equivalence classes are determined by the equivalence classes of definite quadratic forms, the author considers only definite forms, that is those for which $f(x, x)\neq 0$ if $x\neq 0$. Let K be the residue class field A/P where A is the ring of integers of K (those elements of K whose values are not greater than 1) and P the principal ideal, (π), formed by the elements of K whose valuation is less than 1. If E has a basis e_i $(1\leq i\leq n)$, and $x=\sum_i \xi_i e_i$, define $\|x\|=\max(|\xi_i|)$ and M_i the set of vectors x in E such that $\|x\|\leq\pi^i$. Let $\bar{E}_i=M_{i-1}/M_i$. Denote by η the canonical homomorphism of A onto $\bar{K}$ and by φ_i $(i=1, 2)$ the canonical homomorphism of M_{i-1} onto $\bar{E}_i$. The following two quadratic forms on $\bar{E}_i$ $(i=1, 2)$ are called the residue class forms of f:

$$f_1(\varphi_1(x), \varphi_1(x))=\eta(f(x, x)) \quad (x\in M_0),$$
$$f_2(\varphi_2(x), \varphi_2(x))=\eta(\pi^{-1}f(x, x)) \quad (x\in M_1).$$

The author proves: 1. Let f and f' be two definite quadratic forms on the vector spaces E and E' with the same dimension over K. Suppose that the residue class forms f_i and f_i' are equivalent for $i=1, 2$. Then f and f' are equivalent. 2. Suppose that f_1 and f_2 are two definite quadratic forms over $\bar{K}$. Then there exists a quadratic form f over K whose residue class forms are equivalent to f_1 and f_2. 3. Two definite forms in 4 variables over a field K with finite residue class field are equivalent.

B. W. Jones (Boulder, Colo.).

Kuniyoshi, Hideo. On purely-transcendency of a certain field. Tôhoku Math. J. (2) **6**, 101–108 (1954).

Let k be a field of characteristic $p>0$ and $K=k(s_1, \cdots, s_{p^n})$ a purely transcendental extension field over k, σ the automorphism of K induced by a cyclic permutation: $s_i\to s_{i-1}$ $(i \bmod p^n)$. Then it is proved that the fixed field L of σ is also a purely transcendental extension field over k. For the proof a system of algebraic independent generators of L/k are constructed with the help of Witt's vectors. *H. Bergström* (Göteborg).

Deuring, Max. Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. II. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1955**, 13–42 (1955).

In part I [same Nachr. **1953**, 85–94; MR **15**, 779] the author defined the zeta-function of a curve C over the number field k by means of

$$(*) \qquad \zeta(s, C, k)=\prod_{\mathfrak{p}}{}' \zeta(s, C, k, \mathfrak{p}).$$

Let C be defined by $f(x, y)=0$. The prime in the right member of (*) indicates that the product is restricted to prime divisors $\mathfrak{p}$ such that the curve $C/\mathfrak{p}$ defined by $f(x, y)\equiv 0 \pmod{\mathfrak{p}}$ is absolutely irreducible and has the same genus as C; moreover, for such $\mathfrak{p}$, $\zeta(s, C, k, \mathfrak{p})$ denotes the zeta-function of the field of algebraic functions on $C/\mathfrak{p}$. The principal result of I states that if C is of genus 1, admits of complex multiplication and contains at least one rational point relative to k, then

$$(**) \qquad \zeta(s, C, k)=$$
$$=\zeta(s, k)\zeta(s-1, k)L_0(s-\tfrac{1}{2}, k, \chi)^{-1}L_0(s-\tfrac{1}{2}, k, \bar{\chi})^{-1},$$

where $\zeta(s, k)$ is the zeta-function of k, χ is a primitive „Grössencharakter" of k and $L_0(s, k, \chi)$ is "essentially" the L-series $L(s, k, \chi)=\prod\{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}\}^{-1}$.

The writer now raises the question whether the definition of $\zeta(s, C, k)$ can be altered in such a manner that that in (**) the L_0's will be replaced by L's. Let $K=k(x, y)$, the field of algebraic functions on C, K of genus 1. Assume that the extension $\mathfrak{p}^*$ of $\mathfrak{p}$ is such that if u is a non-constant function of K then $u/\mathfrak{p}^*$ is not constant, secondly the field of constants of $K/\mathfrak{p}^*$ is identical with $k/\mathfrak{p}$, and finally the genus of $K/\mathfrak{p}^*$ is 1. Then the extension is called regular and u is $\mathfrak{p}$-regular. It is proved that a given $\mathfrak{p}$ has either a uniquely determined extension (which is independent of u) or it has none; the proof is simple for $\mathfrak{p}\nmid 2$, less simple for $\mathfrak{p}|2$. Now define

$$\zeta(s, K)=\prod_{\mathfrak{p}}\zeta(s, K, \mathfrak{p}),$$

where the product is over all prime divisors of k and $\zeta(s, K, \mathfrak{p})$ is the zeta-function of $K/\mathfrak{p}$ for regular $\mathfrak{p}$ while for irregular $\mathfrak{p}$ it is the zeta-function of $k/\mathfrak{p}$ of genus 0. Then we have the following main result:

$$(***) \quad \zeta(s, K)=\zeta(s, k)\zeta(s-1, k)L(s-\tfrac{1}{2}, k, \chi)^{-1}L(s-\tfrac{1}{2}, k, \bar{\chi})^{-1},$$

where L denotes a Hecke L-series and moreover the conductor of χ contains just the irregular $\mathfrak{p}$. The proof will be completed in a later paper. *L. Carlitz.*

Carlitz, Leonard. Some class number relations. Math. Z. **62**, 167–170 (1955).

From Kronecker's and Hurwitz's class-number relations some other class-number relations are derived.

H. Bergström (Göteborg).

Diophantine Approximation

Poitou, Georges. Approximations diophantiennes et groupe modulaire. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 15–21 (1955).

An exposition of results which can be obtained directly from geometric considerations relative to the modular and Picard groups. The results concern the minimum of a positive definite quadratic or Hermitian form, and the approximation of a number by fractions (rational, or from an algebraic number field having only principal ideals), the latter problem also being modified by requiring that the denominators of the fractions be not divisible by a given integer. *W. J. LeVeque.*

Chalk, J. H. H. Rational approximations in the complex plane. J. London Math. Soc. **30**, 327–343 (1955).

Let $f(u, v)=C(u-\Omega v)(u-\Omega' v)$ be a form of discriminant $D\neq 0$, the coefficients C, Ω, Ω' are arbitrary complex numbers. Let $H(u, v)=auu^*+buv^*+b^*u^*v+cvv^*$ (* denotes complex conjugate) be an indefinite

hermitian form (a and c real, $\Delta=b\bar{b}-ac>0$). Perron [Math. Z. 35, 563–578 (1932)] showed the existence of infinitely many solutions of $|f(u, v)|\leq(\frac{1}{2}|D|)^{\frac{1}{2}}$ in gaussian integers u, v. Oppenheim [Proc. Nat. Acad. Sci. U.S.A. 15, 724–727 (1929)] obtained a result on real indefinite quartic forms which can be specialized to the fact that $|H(u, v)|\leq(\frac{3}{5}\Delta)^{\frac{1}{2}}$ has a solution. Subsequently, it was shown that there are infinitely many solutions in this case also.

The present author obtains these results by modification of a method of Ford [Trans. Amer. Math. Soc. 27, 146–154 (1925)]; a central rôle is played by the Picard group [cf. Ford, ibid. 19, 1–42 (1918)]. The author's explicit results are too long to be quoted here.

N. G. de Bruijn (Amsterdam).

Gonçalves, J. Vicente. Sur le développement des irrationnalités quadratiques en fraction continue. Univ. Lisboa. Revista Fac. Ci. A. (2) 4, 273–282 (1955).

Every square root of a positive number which is not a perfect square can be written $\xi=(p+\delta)/q$, where $\delta=\sqrt{D}$, and where p, q, $D>0$, and $(D-p^2)/q$ are integers. Let an infinite sequence of values $a_n=\pm 1$ be given, b_0 an integer between ξ and $\xi-a_1$ so that $\xi=b_0+a_1/\xi_1$, $\xi_1>1$, b_1 an integer between ξ_1 and ξ_1-a_2 so that $\xi_1=b_1+a_2/\xi_2$, $\xi_2>1$, etc. In general, $\xi_n=(p_n+\delta)/q_n$ with the recurrence relations $p_{n-1}+p_n=b_{n-1}q_{n-1}$, $D-p_n^2=a_nq_{n-1}q_n$, where the p_i and q_i are integers. Let

$$b_0+\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots$$

be a semi-regular continued fraction, i.e., $a_{n+1}+b_n\geq 1$, $n>0$, and $b_n>2$. Then Perron [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954, 321–333; MR 16, 1090] showed that: (i) if $a_n=1$ for and only for $n=n_1, n_2, \cdots, n_i<n_{i+1}$, then (A) $|p_{n_{k+1}}|<\max(\delta, |p_{n_k}|)$; if beyond a certain order, $a_n=-1$, and if $b_n>2$ for and only for $n=n_1, n_2, \cdots$, then (B) $|p_{n_{k+1}}|<\max(3\delta, |p_{n_k}|)$. In this paper it is shown that inequalities (A) and (B) can be obtained from a formula which is derived here. Furthermore, an extension of this new formula to real fractions of a more general type is indicated.

E. Frank (Chicago, Ill.).

ANALYSIS

★**Yaglom, A. M., i Yaglom, I. M. Neèlementarnye zadači v èlementarnom izloženii. [Non-elementary problems in an elementary exposition.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 543 pp. 10.15 rubles.

This book is divided into three parts. The first part contains 170 problems. The second part, about four-fifths of the book, contains the details of the solutions. The third part contains the answers, where answers are required, and short hints. This organization allows the student to adjust his use of the book to his ability and independence. The problems vary in difficulty from simple counting problems requiring only ingenuity to problems requiring considerable mathematical maturity for their solution. Difficult problems are indicated with one, two, or three stars, in order of difficulty. For example, the derivation of Stirling's formula for $n!$ is a three-star problem. There are two sections of problems. The first contains combinatorial problems and probability problems. The second (harder) section contains miscellaneous problems, involving relations between points and lines, topology, evaluation of infinite series, number theory, and so on. The book should be a useful source of problems for teachers and students.

J. L. Doob (Urbana, Ill.).

★**Germay, R. H. J. Cours d'analyse supérieure de la Licence en Sciences mathématiques. Vol. I. Théorie des fonctions (en 2 fascicules).** Université de Liège, Liège, undated. pp. 1–126, 287–732.

★**Germay, R. H. J. Cours d'analyse supérieure de la Licence en Sciences mathématiques. Vol. II. Equations différentielles (en 3 fascicules).** Université de Liège, Liège, 1952. 1027+xiii+xi pp.

This lithoprinted manuscript contains the portion of the author's "Cours" that was available at his death. Fasc. 1 of Vol. I assumes the elementary theory of sequences and series and treats infinite series and products and Riemann-Darboux integration. Fasc. 2 contains chapters on general properties of holomorphic functions, development in series of polynomials, sequences of holomorphic functions, and analytic continuation. The first two fascicules of vol. II are devoted to ordinary differential equations. There is an unusually detailed discussion of the method of successive approximations, and much attention is devoted to normal systems, especially in the linear case. There is a chapter on the complex domain and one on linear systems with periodic coefficients, but very little on the topology of solution curves. The bulk of the third fascicule concerns first-order partial differential equations, a field which occupied much of the author's professional interest. Although the chapter headings are familiar, there are many novel details in the exposition. An extensive addendum, drawn from the author's dissertation, gives a Fuchsian theory for a system of first-order linear ordinary differential equations similar to that presented earlier for a single equation of higher order. The extremely detailed nature of much of the exposition is explained by the stated aim of sparing beginning students the obligation of taking notes at the expense of attention to an oral explanation.

F. A. Ficken (Knoxville, Tenn.).

Hartman, Philip and Wintner, Aurel. Mean value theorems and linear operators. Amer. Math. Monthly 62, 217–222 (1955).

The method of proof by which the mean-value theorem of the differential calculus follows from Rolle's theorem is given a general formulation. This is shown to yield simple proofs of Pólya's generalization of the mean-value theorem [Trans. Amer. Math. Soc. 24, 312–324 (1924)], a theorem of Zaremba on harmonic functions [S. Zaremba, Conférences Réunion Internat. Math., Paris, 1937, Gauthier-Villars, Paris, 1939, pp. 171–176], two mean-value theorems of Blaschke [Jber. Deutsch. Math. Verein. 27, Abt. 1, 157–160 (1918)], and other similar results involving elliptic or parabolic linear differential operators. In some cases, unnecessary restrictions are discarded. The reviewer does not follow the proof that (iii tre) implies (iii), the difficulty lying in the possible discontinuity of $L(u)$. In the applications to differential (and generalized differential) operators, this difficulty is easily overcome. There are some unimportant misprints.

F. F. Bonsall (Newcastle-on-Tyne).

Rogosinski, H. P. General means for finite sets of real numbers. J. London Math. Soc. **30**, 449–463 (1955).

For a given set of real numbers $x_1 \leq \cdots \leq x_n$, and for real functions $f(x, y)$, $g(x, y)$, the author replaces x_1 and x_n by $f(x_1, x_n)$ and $g(x_1, x_n)$, reorders the set, and repeats the process indefinitely. It is shown that, under mild restrictions of a mean-value type on the functions f, g, all the n numbers of the mth set formed by the process will tend to a common limit as $m \to \infty$. For example, $f=g=\frac{1}{2}(x_1+x_n)$ leads to the arithmetic mean; and, for $x_i \geq 0$, $f=(x_1x_n)^{\frac{1}{2}}$, $g=\frac{1}{2}(x_1+x_n)$ lead to the arithmetic-geometric mean. Simple (quasi-linear), reciprocal, and homogeneous means are discussed, and a consistency result is established for applications to infinite sequences (x_i). *E. F. Beckenbach* (Los Angeles, Calif.).

Fan, Ky, Taussky, Olga, and Todd, John. Discrete analogs of inequalities of Wirtinger. Monatsh. Math. **59**, 73–90 (1955).

The following are typical of the inequalities of the title. (1) If $x(0)=0$, then $\int_0^{\pi/2} x^2dt < \int_0^{\pi/2} (x')^2dt$ unless $x=A \sin t$. (2) If $x(0)=x(\pi)=0$, and if $x' \in L^2$, then $\int_0^{\pi} x^2dt < \int_0^{\pi} (x')^2dt$ unless $x=A \sin t$. (3) If $x'(0)=x'(\pi)=0$, if $\int_0^{\pi} x dt=0$, and if $x' \in L^2$, then $\int_0^{\pi} x^2dt < \int_0^{\pi} (x')^2dt$ unless $x=A \cos t$. While (1) is in Hardy, Littlewood, and Pólya, "Inequalities" [Cambridge, 1st ed., 1934, 2nd ed., 1952; MR 13, 727], (2) and (3) are presumably new.

Using standard tools on the values of $(Hx, x)/(x, x)$ for finite Hermitian matrices H, the authors prove discrete analogs of results (1)-(3) and related results. The analog of (2), for example, is the following theorem: (4) If $x_1, \cdots, x_n$ are real numbers, then

$$\sum_{i=0}^{n-1} (x_i - 2x_{i+1} + x_{i+2})^2 > 16 \sin^4 \{\pi(2n+2)^{-1}\} \sum_{i=0}^{n} x_i^2,$$

(where $x_0=x_{n+1}=0$), unless $x_i=A\hat{x}_i$, where $\hat{x}_i = \sin\{i\pi(n+1)^{-1}\}$. As with all the analogs, result (2) (without the equality clause) follows from (4) by a passage to the limit. Similar analogs are given for integral inequalities of R. Bellman [J. London Math. Soc. **18**, 140–142 (1943); MR **5**, 230] and D. G. Northcott [ibid. **14**, 198–202 (1939); MR **1**, 71]. Hölder's inequality is a major tool here.

G. E. Forsythe (New York, N.Y.).

Benedetti, Carlo. Del massimo valore dell'indice di oscillazione in una successione di termini al variare in tutti i modi possibili l'ordine di questi. Metron **17**, no. 3–4, 53–60 (1955).

Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive numbers. The author determines the maximum which the oscillation index $\sum_1^{n-1} |a_i - a_{i+1}|/(n-1)$ can assume under permutations of the elements of the sequence.

E. Lukacs (Washington, D.C.).

Benedetti, Carlo. Sul massimo e sul minimo dell'indice semplice di dissomiglianza. Metron **17**, no. 3–4. 61–65 (1955).

The author considers two finite sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ each containing n real numbers. He determines the maximum and the minimum of the sum $\sum_{i=1}^{n} |a_i - b_i|$ under the following conditions: a) both sequences are nondecreasing and contain only non-negative numbers; b) the means of both sequences are fixed. *E. Lukacs* (Washington, D.C.).

Backes, F. Sur une formule de Dirichlet et le reste de Laplace. Mathesis **64**, 5–8 (1955).

By repeated applications of the Dirichlet formula

$$\int_0^x dt \int_0^t f(z, t)dz = \int_0^x dz \int_z^x f(z, t)dt,$$

the author obtains the well-known expression

$$(*) \qquad \Phi_n(x) = \frac{1}{(n-1)!} \int_a^x f(z)(x-z)^{n-1}dz$$

for the function $\Phi_n(x)$ which, together with its first $n-1$ derivatives, vanishes at the origin, and whose nth derivative is equal to $f(x)$. We note that the expression might equally well have been obtained through integration by parts. Application is made to the derivation of the Laplace formula for the remainder in the finite Taylor series, and an expression analogous to (*) is obtained for a function of several variables, one of whose mixed partial derivatives is given.

E. F. Beckenbach (Los Angeles, Calif.).

★ **Albrecht, Rudolf und Hochmuth, Hans. Übungsaufgaben zur höheren Mathematik. Teil II.** Verlag von R. Oldenbourg, München, 1955. 131 pp. DM 9.80.

The problems are grouped under the following headings: Determinanten, lineare Gleichungen; komplexe Zahlen; Vektoren, analytische Geometrie des Raumes; Funktionen von mehreren Veränderlichen; Flächen und Raumkurven; Mehrfache Integrale; Integralsätze der Vektorrechnung.

Theory of Sets, Functions of Real Variables

Hiraguchi, Toshio. On the dimension of partially ordered sets. Sci. Rep. Kanazawa Univ. **1**, 77–94 (1951).

The dimension alluded to is that of Dushnik and Miller [Amer. J. Math. **63**, 600–610 (1941); MR **3**, 73]. Let $D[P]$ and $n[P]$ be the dimension and power respectively of the partially ordered set P. The main result obtained is a generalization of the theorem of Dushnik and Miller which states that a partially ordered set of dimension n can be defined on a set of power $2n$, namely: If $n[P] \geq 4$, then $D[P] \leq [n[P]/2]$, where the latter symbol means the integer part of $n[P]/2$ when $n[P]$ is finite and $n[P]$ itself when $n[P]$ is transfinite. The proof is by induction together with a case analysis. Another result obtained is on linear extensions. A linear extension $L[C, P]$ of a partially ordered set P is said to be left with respect to a subset C provided that, if x is incomparable with z, z in C, then $z \geq x$. Then $L[C, P]$ is non-empty if and only if C is simply ordered. Finally the author discusses the relationship between $D[P]$ and $D[Q]$ for Q a subset of P. If C is simply ordered and $C=P-Q$, then $D[P] \leq D[Q]+2$. An example is given to show that this result cannot be improved to $D[Q]+1$. Certain sufficiency conditions are also given on $P-Q$ in order that $D[P] \leq D[Q]+1$. For example, if a is a minimal and b a maximal element of a finite P and if a is incomparable with b in P, then $D[P] \leq D[Q]+1$, where $Q=P-\{a, b\}$. [Actually the result holds true even if P is not finite.] There are numerous misprints, the most serious being the following. On page 79, line 16, read C for A; on page 82, line 20, for (a, b) read (a, b'); on page 86, line 38, for $x>a$, $x>b$, read $x>a$, x incomparable with b; and on page 87, in the diagram for P_3^* read c' for d' and d' for c'.

S. Ginsburg (Inglewood, Calif.).

Ohkuma, Tadashi. Sur quelques ensembles ordonnés linéairement. Proc. Japan Acad. **30**, 805–808 (1954).

L'auteur étudie les chaînes qui présentent la propriété suivante (P): A toute paire a, b d'éléments de la chaîne E, il existe un automorphisme $f(x)$ et un seul de E pour lequel $f(a)=b$. Les résultats suivants sont signalés: Il existe une chaîne jouissant de la propriété (P) qui n'est pas semblable à la chaîne de tous les nombres entiers. L'ensemble de tous les types différents des chaînes possédant la propriété (P) a la puissance $2^{2^{\aleph_0}}$.

M. Novotný (Brno).

Ginsburg, Seymour. Order types and similarity transformations. Trans. Amer. Math. Soc. **79**, 341–361 (1955).

This paper is closely connected with three other papers of the author [same Trans. **74**, 514–535 (1953); **77**, 122–150 (1954); Proc. Amer. Math. Soc. **5**, 554–565 (1954); MR **14**, 853; **16**, 20, 21]. Here, in particular, the "P-problem" is examined in order to find out whether there exists an order type τ such that $\sigma<\tau<\mu$ (σ, μ being two given order types). E.g. if E is a linear set of power c, then the P-problem applied to $\sigma=0$, $\mu=tE$, admits of a solution which has the A-property [which is exact and of power c] (Th. 5) [Th. 4]. Let α be any linear order type so that $\alpha<\lambda$ [resp. $|\alpha|=c$]; let Y be a set of power c and H a $\leq$-chain of sets $\leq Y$. Then the P-problem for $\sigma=\alpha$, $\mu=\lambda$ [resp. $\sigma=0$, $\mu=\alpha$] admits of a set, similar to H, of exact solutions, each of power c (Th. 6 [resp. Th. 7]) and there exists a set of power $>c$ of exact solutions, of power c each (Th. 10) [resp. Th. 11]. If the chain A has two fixed points p, q so that the interval $(p, q)_A$ is infinite, then there exists a schism of $(t(A-\{p\}), tA)$ and a schism of $(t(A-\{q\}), tA)$ (Th. 18); one has $t(A-\{p\})\|t(A-\{q\})$ (Th. 19). Definition. Let A, B be chains so that $tA<tB$; if $A=F+G$, $tF+1+tG\|tB$; then the ordered pair (tF, tG) is called a schism of (tA, tB). As to other terminology, see the cited reviews. *G. Kurepa* (Zagreb).

Albuquerque, J. Le successioni di insiemi. Univ. Lisboa. Rev. Fac. Ci. A. (2) **4**, 201–224 (1955).

L'A. étudie les opérations ps, sp que voici: E_n $(n<\omega)$ étant une suite d'ensembles, l'A. définit psE_n comme l'intersection des ensembles $E_{k_0}\cup E_{k_1}\cup\cdots$; ici $k_0, k_1, \cdots$ parcourt le système de toutes les suites infinies d'entiers positifs. D'une façon duale on définit spE_n. On a $CpsE_n=spCE_n$ (Th. 7). L'égalité $psE_n=spE_n$ a lieu si et seulement si la suite E_n est convergente au sens de de la Vallée-Poussin (Th. 16); la même égalité est caractérisée par le système que voici: $psE_n\cap psCE_n=0$, $psE_n\cup psCE_n=1$ (Th. 14). La suite-réunion (la suite-intersection) d'une suite de suites pareilles est encore une suite pareille (Th. 19). Pour chaque suite E_n on a (Th. 20)

$$\liminf E_n\leq spE_n\leq psE_n\leq \limsup E_n.$$

G. Kurepa (Zagreb).

Boboc, N. Un exemple de fonction continue de Darboux, discontinue partout. Com. Acad. R. P. Romîne **4**, 199–200 (1954). (Romanian. Russian and French summaries)

Marcus, S. La limite approximative qualitative. Com. Acad. R. P. Române **3**, 9–12 (1953). (Romanian. Russian and French summaries)

Define the upper qualitative limit, $\mathrm{lqs}_{x\to x_0} f(x)$ to be the greatest lower bound of numbers y for which $\{x|f(x)>y\}$ is of first category at x_0, and let $\mathrm{lqi}_{x\to x_0} f(x)=-\mathrm{lqs}_{x\to x_0}-f(x)$: when these are equal, the author calls the common value the qualitative limit of $f(x)$ at x_0. He states various properties of this operation which resemble those of the approximate limit of Denjoy and Hinčin.

M. M. Day (Urbana, Ill.).

Marcus, S. La continuité approximative qualitative. Com. Acad. R. P. Române **3**, 117–120 (1953). (Romanian. Russian and French summaries)

Qualitative continuity is defined in terms of the qualitative limit of the preceding review. The author gives theorems connecting this with linear continuity of domains in the plane, and with the property of Baire for functions of a metric space. *M. M. Day* (Urbana, Ill.).

Marcus, S. Sur un théorème de G. P. Tolstov. Com. Acad. R. P. Române **2**, 5–8 (1952). (Romanian. Russian and French summaries)

Tolstov's theorem says: If $f(x, y)$ is linearly continuous in a domain G and if the partial derivative with respect to x exists for all except possibly a countable set E of points of G, then every perfect set P contained in G contains a portion on which f is continuous (in both variables). The present note shows also that (1) the closure of the set of points of discontinuity of f is non-dense, (2) the set E of exceptional points is also non-dense, and (3) even if E is only assumed to be non-dense and F_σ, every interval intersecting G contains a subinterval of points where f is continuous. *M. M. Day* (Urbana, Ill.).

Marcus, S. Sur les discontinuités des fonctions de trois ou plusieurs variables, à nombres dérivés partiels continus. Com. Acad. R. P. Române **2**, 125–128 (1952). (Romanian. Russian and French summaries)

The author generalizes to three variables a result of Tolstov [Izv. Akad. Nauk SSR. Ser. Mat. **13**, 425–446 (1949); MR **11**, 167] on the size of the set described in the title. *M. M. Day* (Urbana, Ill.).

Nicolescu, Miron. Extensions de la notion de continuité pour les fonctions de plusieurs variables. Com. Acad. R. P. Române **2**, 9–11 (1952). (Romanian. Russian and French summaries)

For $n=1$, 2, or 3, and $1\leq i\leq n$ the author defines for each function $u(x)$ of n variables $x=(x_1, \cdots, x_n)$ an i-dimensional difference. In the familiar notation, which is not that of the author, let Δ_i and ∇_i be the forward and backward differences on the ith variable,

$$\Delta_i u(x)=u(\cdots, x_i+\Delta x_i, \cdots)-u(\cdots, x_i, \cdots)$$

and

$$\nabla_i u(x)=u(\cdots, x_i-\Delta x_i, \cdots)-u(\cdots, x_i, \cdots).$$

Let $\Delta_{ij}u(x)=\Delta_i\Delta_j u(x)$ $(=\Delta_j\Delta_i u(x))$ and let $\Delta_{ijk}=\Delta_i\Delta_j\Delta_k$; make similar definitions for ∇_{ij} and ∇_{ijk}. Setting $\Delta u(x)=u(x+\Delta x)-u(x)$, the author observes that if $n=2$, then $2\Delta_{12}u(x)=-(\nabla_1+\nabla_2)u(x+\Delta x)-(\Delta_1+\Delta_2)u(x)$, and $2\Delta u(x)=-(\nabla_1+\nabla_2)u(x+\Delta x)+(\Delta_1+\Delta_2)u(x)$. Also, when $n=3$,

$$3\Delta_{123}u(x)=(\nabla_{12}+\nabla_{23}+\nabla_{31})u(x+\Delta x)-(\Delta_{12}+\Delta_{23}+\Delta_{31})u(x).$$

(3 is misprinted 2 in the paper). Analogy suggests using + instead of − to define $2\Delta^2u(x)$. The author calls a function u of three variables di(tri)dimensionally continuous at x if $\Delta^2u(x)$ $(\Delta_{123}u(x))$ goes to zero with Δx, and observes that this property is preserved by sums and constant multiples. The author makes no reference to the considerable body

of literature on differences; he does not mention, for example, that the functions $\Delta_{12\ldots n}u$ is the interval function used in computing a Lebesgue-Stieltjes measure from u. His notation is so inconvenient that he can not observe, for example, that $2\Delta_{12}u(x)$ and $2\Delta^2u(x)$ are respectively the cases $n=2$ and $n=3$ of

$$-\Sigma_{i\leq n}(\Delta_i u_i(x)+\nabla_i u(x+\Delta x)).$$

M. M. Day (Urbana, Ill.).

Nicolescu, Miron. Les différentielles polydimensionnelles, de différents ordres, des fonctions de plusieurs variables. Com. Acad. R. P. Române **2**, 181–184 (1952). (Romanian. Russian and French summaries)

If u is a function of n variables, $n=2$ or 3, u is called hyperbolic differentiable at x if there exist $A(x)$ and $\sigma(x, \Delta x)$ such that $\Delta_{1\ldots n}u(x)=(A+\sigma)\Delta x_1\cdots\Delta x_n$ and $\sigma\to 0$ when $\Delta x\to 0$. A function u of 3 variables is called bidimensionally differentiable at x if there exist $A_i(x)$ and $\sigma_i(x, \Delta x)$ $(i=1, 2, 3)$, such that all $\sigma_i\to 0$ when $x\to 0$ and

$$\Delta^2u(x)$$
$$=(A_3+\sigma_3)\Delta x_1\Delta x_2+(A_1+\sigma_1)\Delta x_2\Delta x_3+(A_2+\sigma_2)\Delta x_3\Delta x_1.$$

Then if u is bidimensionally differentiable at a point x, it is bidimensionally continuous at x, and x is 2-dimensiona hyperbolic differentiable there.

M. M. Day (*Urbana*, Ill.).

Menger, Karl, and Shü, S. S. Generalized derivatives and expansions. Proc. Nat. Acad. Sci. U.S.A. **41**, 591–595 (1955).

B: system of n (real or complex) numbers, not necessarily distinct. f: function having the necessary properties of definition and differentiability at the points of B. $f[B]$: the interpolation polynomial of f at B. § I. B: one-parameter family of n-tuples $B=B(t)$. $s_1(t), \cdots, s_n(t)$: the elementary symmetric functions of the numbers in $B(t)$. f: function such that $f[B(t)]$ is defined for every t in a neighborhood of t_0. The derivative $D_{B,t_0}f$ at t_0 of f with respect to the family B is defined as the polynomial

$$\lim_{t\to t_0}\frac{f[B(t)]-f[B(t_0)]}{t-t_0}.$$

The mth order derivative $D^m_{B,t_0}f$ is defined inductively. Assuming that $s_1, \cdots, s_n$ have classical derivatives of mth order at t_0: sufficient conditions are given for the existence of $D^m_{B,t_0}f$ in the complex and real case; $D_{B,t_0}f(z)$ is represented as a contour integral in the complex case; $f[B(t)]$ is expressed by a generalized finite Taylor expansion with remainder. § II. g is a complex function and A a finite system of (not necessarily distinct) numbers such that each point a of A has a neighborhood U_a in which g is analytic without being identical with $g[A]$. For any number t, $B(t)$ denotes the system of those roots (each counted with its multiplicity) of the equation $g(z)-g[A](z)=t$ lying in the union of the U_a. There exist two positive numbers d and e such that, for $|t|\leq e$ and $U_a: |z-a|\leq d$, all systems $B(t)$ contain the same number of points; their family is denoted by $B\{g, A\}$. The polynomial $D_{B\{g,A\},0}f$, when it exists, is called the derivative of f with respect to g at the system A, in short, $D_gf(A)$. Infinite expansions for f are given involving such derivatives; they include as special cases the expansions of Taylor, Lagrange, Jacobi, Bürmann, Laurent, Teixera and Kienast. The note contains no proofs.

C. Pauc (Nantes).

Jacobsthal, Ernst. Beiträge zur Differentialrechnung. I. Norske Vid. Selsk. Forh., Trondheim **27** (1954), 146–150 (1955).

Let $f(x)$ and $g(x)$ be repeatedly differentiable and let $F(x)=g(x)/f(x)$. The calculation of the nth derivative $F^{(n)}(x)$ by the recurrence method for large values of n is a tedious task. A representation of $F^{(n)}(x)$ is obtained which minimizes the amount of work involved. One of the relations involving $F^{(n)}$ is

$$f^{n+1}F^{(n)}=\sum_{\varrho=0}^{n}\alpha_{n\varrho}f^{n-\varrho},\quad \alpha_{n0}=\sum_{\lambda=0}^{\varrho}(-1)^\lambda\binom{h}{\varrho}f^{\lambda-\varrho}(f^\lambda\cdot g)^{(n)}.$$

It is then shown that $\alpha_{n\varrho}$ can be obtained by omitting from the expansion of $(-1)^\varrho(f^n\cdot g)^{(n)}$ all members containing f itself. This is denoted by $(-1)^\varrho(f^n\cdot g)_0{}^{(n)}$. Then

$$f^{n+1}F^{(n)}=\sum_{\varrho=0}^{n}(-1)^\varrho f^{n-\varrho}(f^n\cdot g)_0{}^n.$$

R. L. Jeffery (Kingston, Ont.).

Theory of Measure and Integration

Götz, A. On the equivalents of the notion of point function in Boolean fields. Prace Mat. **1**, 145–161 (1955). (Polish. Russian and English summaries)

A 1-to-1 correspondence is established between "functionoids" studied by J. M. H. Olmsted [Trans. Amer. Math. Soc. **51**, 164–193 (1942); MR **4**, 11] and by C. Carathéodory [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1938**, 27–69] so that corresponding functionoids have the following properties: (a) they have equal integrals with respect to every measure; (b) they determine the same measurable point-function if the elements of the underlying σ-complete Boolean algebra are subsets of a set. This correspondence still holds if Olmsted's functionoids are replaced by their extension due to E. Marczewski. *H. M. Schaerf* (St. Louis, Mo.).

Srinivasan, T. P. On extensions of measures. J. Indian Math. Soc. (N.S.) **19**, 31–60 (1955).

This is a technical study of extension theorems, old and new, and of the feasibility of using inner measure instead of outer measure in the proofs of such theorems. Two methods of extending linear functionals to full-fledged integrals (metric completion and order completion) are adapted to a set-theoretic (as opposed to linear) context and are shown to yield the same result. A by-product is a proof of the Hahn-Kolmogoroff extension theorem in which existence and uniqueness are obtained simultaneously. Another application yields a re-derivation of the known facts concerning the construction of Borel measures on locally compact Hausdorff spaces.

P. R. Halmos (Chicago, Ill.).

Phakadze, Š. S. On various definitions of the concept of sets absolutely of measure zero. Soobšč. Akad. Nauk Gruzin. SSR **15**, 489–496 (1954). (Russian)

The author studies properties of and implications among four possible definitions of the concept mentioned in the title. The three new ones [cf. same Soobšč. **15**, 201–205

(1954); MR 16, 1092] turn out to be such that the class of sets in question is not closed under the formation of finite unions. *P. R. Halmos* (Chicago, Ill.).

Bahadur, R. R. Statistics and subfields. Ann. Math. Statist. **26**, 490–497 (1955).

Let X be a Borel set of a Euclidean space, S be the σ-field of Borel sets of X, and μ be a probability set function on S. A σ-field $S_0 \subseteq S$ is called a subfield. Let f be any (not necessarily real-valued) function on X and define S_f, "the subfield induced by f", as the class of all sets in S which are of the form $f^{-1}(B)$, where B is a subset of the range of f. Theorem 1: Every subfield is inducible by some function f if and only if X is countable.

A subfield S_1 is a "contraction" of a subfield S_2 if, for each S_1-measurable real-valued function $f_1(x)$ on X there exists an S_2-measurable function $f_2(x)$ such that $f_1(x)=f_2(x)$ except on a set of μ-measure zero. S_1 and S_2 are "equivalent" if each is a contraction of the other. Theorem 2: For any subfield S_0 there exists an f such that S_f is equivalent to S_0. This f may be taken to be real-valued and S-measurable.

A function f is defined to be a "contraction" of a function g if S_f is a contraction of S_g; f is defined to be a "functional contraction" of g if there exists a function h such that $f(x)=h(g(x))$ except on a set of μ-measure zero. The two notions are shown to be in general different. Theorem 3: If both f and g are real-valued and S- measurable, the two notions are the same. *J. Wolfowitz.*

Goffman, Casper. Convergence in area of integral means. Amer. J. Math. **77**, 563–574 (1955).

§ 1. $x(t)$: (Lebesgue) summable function on $[0, 1]$, $x_n(t)$: the integral mean $n\int_0^{1/n} x(t+u)du$ defined on $[0, 1-1/n]$. The generalized variation $\Phi(x)$ was defined by the author in a previous paper [Rend. Circ. Mat. Palermo (2) **2**, 203–235 (1954); MR **16**, 457] like the ordinary variation but using as subdivision points only points of approximate continuity of $x(t)$. Lemma: $\lim \Phi(x_n)=\Phi(x)$. Theorem 1: If $x(t)$, $y(t)$ are summable functions defined on $[0, 1]$ and C_n is the curve $x=x_n(t)$, $y=y_n(t)$, then the sequence of the lengths $L(C_n)$ converges on $[0, +\infty]$. § 2. $f(x, y)$: summable function on the unit square $[0, 0; 1, 1]$; $f_n(x, y)$: the integral mean $n^2\int_0^{1/n}\int_0^{1/n} f(x+u, y+v)\,du\,dv$ defined on $[0, 0; 1-1/n, 1-1/n]$. E_x, E_y: subsets of $[0, 1]$ (of measure 1) such that for every $x \in E_x$, $y \in E_y$, $f(x, y)$ is summable in the other variable. $\Phi(f; x)$: generalized variation of $f(x, y)$ as a function of y for $x \in E_x$. $\Phi_1(f)=\int_{E_x}\Phi(f; x)dx$. Similarly for $\Phi_2(f)$. Theorem 2: $\lim \Phi_1(f_n)=\Phi_1(f)$ and $\lim \Phi_2(f)=\Phi_2(f)$. If $\Phi_1(f)$ and $\Phi_2(f)$ are both finite, the Lebesgue area $A(f_n)$ of the surface $z=f_n(x, y)$ converges to the generalized Geöcze expression $\Psi(f)$, and also to the area $H(f)$ of the paper quoted above. § 3. For the case of parametric surfaces a result of C. B. Morrey concerning the continuous case is extended; the proof is omitted as essentially the same as Morrey's. Remarks by the reviewer: (i) Apart from minor adjustments the functions $f(x, y)$ with $\Phi_1(f)<\infty$ and $\Phi_2(f)<\infty$ are de Giorgi's functions admitting a generalized gradient in the plane [de Giorgi, Ann. Mat. Pura Appl. (4) **36**, 191–213 (1954); MR **15**, 945; see also a paper of the reviewer in the same journal (in press)]. (ii) The interpretation of $\Phi(x)$ as a Burkill integral (as in the continuous case) is not a simple consequence of the definition as asserted on p. 564, lines 18–19. The reviewer has not been able to prove it for $\Phi(x)<\infty$ and has invalidated it for $\Phi(x)=\infty$ by the following counter-example: $x(t)$ is the characteristic function of an open subset of $[0, 1]$ containing the rational numbers and having measure <1. (iii) The measurability of $\Phi(f; x)$ and $\Phi(f; y)$ is not mentioned. *C. Pauc* (Nantes).

Kudryavcev, L. D. On the ϕ-variation of mappings and summability of powers of the Radon-Nikodým derivative. Uspehi Mat. Nauk **10**, no. 2(64), 167–174 (1955). (Russian)

The author gives an abstract form to a well-known necessary and sufficient condition for a function of sets to be the integral of a function $g(x) \in L^p$ where $p>1$; he applies it to define what he terms the pth variation of a mapping f of a space X into a space Y subject to the following hypotheses: (i) There exists a non-negative measurable function $g(x)$ such that the measure of $f(E)$ is the integral of g on E whenever E is a measurable set on which f is one-one and that further; (ii) X is a countable sum of measurable sets $E_0+\sum E_n$ where f is one-one in E_n $(n>0)$ and where the measure of $f(E_0)$ and the integral of g on E_0 vanish. It is shown that the pth variation in question is then the integral of g^p. A change-of-variable formula is established on the further assumption that $f^{-1}(A)$ is measurable whenever A is a measurable set in the space Y. *L. C. Young* (Madison, Wis.).

Weissinger, Johannes. Über einen Differentiationsprozess für Abbildungen und die Transformationsformel für Gebietsintegrale. Wiss. Z. Tech. Hochsch. Dresden **2**, 461–465 (1953).

Es sei $P \to \Phi(P)$ eine Abbildung der Punkte P eines n-dimensionalen Euklidischen Raumes in einen n-dimensionalen Raum hinein. Wenn $\lim J[\Phi(Q_\nu)]/J[Q_\nu]=\Phi'(P)$ für jede sich auf P zusammenziehende Folge von Würfeln Q_ν existiert ($J[\cdots]$ ist der Inhalt der betr. Punktmengen), so heißt $\Phi'(P)$ in Analogie zur gewöhnlichen Ableitung einer Funktion die Maßableitung von Φ. Es gilt ähnlich wie in der gewöhnlichen Differentialrechnung ein Mittelwertsatz, mit dessen Hilfe die Transformationsformel für Gebietsintegrale ohne viele Epsilontik hergeleitet werden kann. Schließlich zeigt man, daß $\Phi'(P)$ im allgemeinen der Absolutbetrag der Funktionaldeterminante der Abbildung ist. *W. Maak* (Zbl **51**, 43).

Kostovskiĭ, A. N. Integrability of continuous surfaces given in polar coordinates. Ukrain. Mat. Ž. **6**, 81–104 (1954). (Russian)

The author considers Lebesgue integrability of a surface given in spherical (polar) coordinates by the equation $\varrho=f(\alpha, \beta)$, where $f(\alpha, \beta)$ is continuous. On the basis of the theory of the Burkill integral and the concept of function of bounded variation in Tonelli's sense [Saks, Theory of the integral, Warsaw-Lwów, 1937, ch. V, § 4], the author proves the following theorem. I. A necessary and sufficient condition for integrability of a continuous surface $\varrho=f(\alpha, \beta)$ in the part belonging to the rectangle

$$D(0 \le \alpha_1 < \alpha \le \alpha_2 < \tfrac{1}{2}\pi;\ 0 \le \beta_1 < \beta \le \beta_2 < 2\pi)$$

is the boundedness of variation in Tonelli's sense of the function $f(\alpha, \beta)$ on the rectangle D. II. If the piece of the surface $S(D)$ belonging to D is integrated, then its area

$$|S(D)| \ge \iint_D f(\alpha, \beta)[f^2(\alpha, \beta)\cos^2\alpha + (\partial f/\partial\alpha)^2\cos^2\alpha + (\partial f/\partial\beta)^2]^{\frac{1}{2}}d\alpha d\beta.$$

III. Let $\Gamma_r \subset D$ be a circle of radius r with center at a point $(\alpha, \beta) \in D$; then

$$\lim_{r\to 0} |S(\Gamma_r)|/\pi r^2 = S'(\alpha, \beta)$$

exists almost everywhere in D and almost everywhere one has

$$S'(\alpha, \beta) = f(\alpha, \beta)[f^2(\alpha, \beta)\cos^2\alpha + (\partial f/\partial\alpha)^2\cos^2\alpha + (\partial f/\partial\beta)^2]^{\frac{1}{2}}.$$

A. S. Kovan'ko (RŽMat **1954**, no. 5503).

Kostovskiĭ, A. N. Expression by a double integral of the area of a surface given in polar coordinates. Ukrain. Mat. Ž. **6**, 398–404 (1954). (Russian)

In this paper, and its (unavailable) predecessor [reviewed above], material from Saks's book cited in the preceding review is transposed to the case of surfaces of the form $r = f(\alpha, \beta)$ where r, α, β are the polar coordinates of a point in Euclidean 3-space. In particular, it is shown that in order that f be of bounded variation in the Tonelli sense and that the area be given by the usual formula in polar coordinates, it is necessary and sufficient that f be absolutely continuous in the Tonelli sense. *L. C. Young.*

Reifenberg, E. R. Parametric surfaces. V. Area. II. Proc. London Math. Soc. (3) **5**, 342–357 (1955).

The purpose of this paper is to establish within the framework of the theory of area developed by A. S. Besicovitch [Proc. Cambridge Philos. Soc. **44**, 313–334 (1948); **45**, 5–23 (1949); J. London Math. Soc. **23**, 241–246 (1948); Quart. J. Math. Oxford Ser. **20**, 1–7 (1949); Nederl. Akad. Wetensch. Proc. Ser. A. **57**, 169–175 (1954); MR **10**, 520, 521; **11**, 167; **15**, 784] and the author [Proc. Cambridge Philos. Soc. **47**, 687–698 (1951); **48**, 46–69 (1952); Quart. J. Math. Oxford Ser. (2) **3**, 227–234 (1952); J. London Math. Soc. **27**, 448–456 (1952); MR **14**, 363, 629] certain basic facts concerning the Lebesgue area and the Hausdorff spherical area for continuous (light) maps of a 2-cell into 3-space. These facts had been previously obtained only from theories developed elsewhere, and perhaps less geometrically. The present contribution serves to make a large part of the author's other papers, including all of his "Parametric surfaces (I)", depend only on the work of Besicovitch, though apparently his "Parametric surfaces (II)" still depends in part (by explicit reference in the proof of Lemma 24) on Cesari's inequality.

First the author indicates a new construction which shows that for a saddle surface (i.e. a surface which bulges across no plane) the Geöcze area is less than or equal to the Hausdorff spherical area. After verifying that this inequality is additive with respect to subdivision by curves of finite length, he infers various properties of general surfaces, including the fact that the Lebesgue area never exceeds the Hausdorff spherical area, and the additivity of the Lebesgue area with respect to subdivision by curves of finite length. *H. Federer.*

Functions of a Complex Variable, Generalizations

Stoïlov, S. Sur quelques aspects modernes de la théorie des fonctions d'une variable complexe. Com. Acad. R. P. Române **1**, 753–756 (1951). (Romanian. Russian and French summaries)

„Il est démontré qu'étant donné un groupe topologique G, localement euclidéen à deux dimensions, simplement connexe et abélien, D un sous-groupe discret de G qui n'est pas engendré par un seul élément, l'homomorphisme h: $G \to G/D$, on peut par un homéomorphisme-isomorphisme et par une transformation intérieure, appliqués à G et à G/D respectivement, transformer h en une fonction méromorphe doublement périodique. Extensions n-dimensionelles basées sur ce genre de caractérisations de classes des fonctions analytiques." (From author's summary.) *M. Heins* (Providence, R.I.).

Farinha, João. Quelques propositions concernant les zéros d'un polynôme. Univ. Lisboa. Revista Fac. Ci. A. (2) **4**, 187–190 (1955).

By associating with the polynomial

$$f(z) = p_0 + p_1 z + \cdots + p_n z^n \quad (p_i \neq 0)$$

the continued fraction

$$f(z) = r_0 + \cfrac{r_1 z}{1} + \cfrac{-r_2 z}{1 + r_2 z} + \cdots + \cfrac{-r_n z}{1 + r_n z},$$

where $r_0 = p_0$, $r_1 = p_1$, and for $1 < i$, $r_i = p_i/p_{i-1}$, the author proves that if, for $1 < i < n$, $\arg r_i = \theta$ and if

$$|\arg(p_1/p_0, r_n) - \theta| \leq \pi/2,$$

then $f(z)$ has no zeros on the line $\arg z = -\theta$. Likewise, by associating with $f(z)$ the continued fraction

$$f'/f = \cfrac{1}{a_1 z + b_1} + \cfrac{1}{a_2 z + b_2} + \cdots + \cfrac{1}{a_n z + b_n},$$

he proves several theorems on the zeros of $f(z)$ including that, if all a_k have the same sign, the zeros of $f(z)$ lie in the strip $A \leq \mathrm{Re}\,(z) \leq B$, where $A = \min(-b_k/a_k)$ and $B = \max(-b_k/a_k)$. *M. Marden* (Milwaukee, Wis.).

Gispert, Hans-Günter. Über die Formeln von Plemelj für Cauchysche Integrale. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **4**, 311–318 (1955).

The author considers a simple, closed, smooth curve S in the complex z-plane and imposes on it the two Liapounoff-conditions. Some properties of such curves are established. On S there is defined a complex-valued function $h(s)$, absolutely integrable on S and possibly satisfying on a sub-arc of S an H (Hölder) condition. He then studies the boundary values of the first-order partial derivatives of the logarithmic potential of density on S. There is given a new proof of the Plemelj-Privalov theorem regarding the H-property of the boundary values, from both sides on S, of Cauchy integrals of density on S. The corresponding known results are established for the first-order partial derivatives of Newtonian potentials of a simple surface-layer. The methods used constitute basically an adaptation of those originally employed by Plemelj in proving his formulas. *W. J. Trjitzinsky* (Urbana, Ill.).

Ricci, Giovanni. Maggiorazione del resto delle serie di potenze sul cerchio di convergenza. Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 121–131 (1955).

This paper deals with the problem of finding an upper bound for the modulus of the nth remainder $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$ of a Taylor series at the point $z = e^{i\theta}$. If the sequence $\{a_k\}$ is of bounded variation and $0 < \theta < 2\pi$, then $|R_n(e^{i\theta})| < \lambda_n/\sin(\theta/2)$, where $\lambda_n = \sum_n^{\infty} |a_k - a_{k+1}|$. If (1) $a_n = n^{-\gamma} e^{i\phi_n}$, where $\gamma > 0$ and $\phi_n - \phi_{n+1} = O(n^{-1})$, then $|R_n(e^{i\theta})| < K/(n^{\gamma} \sin\theta/2)$. One of the author's remarks in this connection is incorrect. It is not true that the Taylor series $\sum a_n z^n$ converges at $z = e^{i\theta}$ $(0 < \theta < 2\pi)$ whenever $0 < \gamma \leq 1$ and a_n is given by (1), with $\phi_n - \phi_{n+1} = o(n^{\gamma-1})$; example: $z = -1$, $\phi_n = (-1)^n n^{\gamma-1}/\log n$.

If $a_n \to 0$ and the function $f(z)=\sum a_n z^n$ is holomorphic on the closed set E on $|z|=1$, then $R_n(e^{i\theta}) \to 0$ uniformly on E. The author proves a general principle which yields an upper bound for $|R_n|$ on E. The following results constitute special cases: If $a_n=O(n^{-\gamma})$ then, for $e^{i\theta}$ in E, $|R_n|<Kn^{-\gamma}$ $(0<\gamma<1)$, $|R_n|<Kn^{-1}\log n$ $(\gamma=1)$, $|R_n|<Kn^{-1}$ $(1<\gamma<2)$, $|R_n|<Kn^{1-\gamma}$ $(2<\gamma)$; in each case, $K=K(E, f)$.

G. Piranian (Ann Arbor, Mich.).

Ricci, Giovanni. Una osservazione sulle funzioni maggioranti delle serie di potenze di norma finita. Boll. Un. Mat. Ital. (3) **10**, 147–153 (1955).

G. H. Hardy [Quart. J. Pure Appl. Math. **44**, 147–160 (1913)] proved that if $f(z)=\sum a_n z^n$ is regular and bounded in the disk $|z|<1$, then $\lim_{r\to 1}(1-r)^{\frac{1}{2}}\bar{M}(f; r)=0$, where $\bar{M}(f; r)=\sum |a_n| r^n$. In the proof of Hardy's theorem all that is required is that $\sum |a_n|^2<\infty$ (which is true if $f(z)$ is bounded in $|z|<1$). The author concerns himself with the class F_2 of functions regular in the unit disk for which $\sum |a_n|^2<\infty$. The following theorem is proven. Let $\phi(t)$ be any real-valued function defined for $t>0$ having the property that $\phi(t)\to+\infty$ as $t\to+\infty$; then there exists an $f \in F_2$ such that

$$\limsup_{r\to 1-} \{(1-r)\phi((1-r)^{-1})\}^{\frac{1}{2}}\bar{M}(f; r)=+\infty.$$

Other related theorems are proven. *V. F. Cowling.*

Skof, Fulvia. Distribuzione di mantisse e serie di potenze non prolungabili. Ann. Mat. Pura Appl. (4) **38**, 303–308 (1955).

Let the radius of convergence of the power series $\sum a_k z^{n_k}$ be one; let the coefficients a_k all lie in the same quadrant of the plane; and let there exist a real irrational number λ such that the sequence $\{\exp(2\pi i n_k \lambda)\}$ converges. Then the unit circle is a natural boundary for the function represented by the power series [J. Gergen and D. V. Widder, Amer. J. Math. **50**, 139–146 (1928)]. The author shows that this theorem remains true if the hypothesis on the position of the coefficients is abandoned and the requirement on the exponents n_k is relaxed to this, that for some irrational number λ the set of numbers $\exp(2\pi i n_k \lambda)$ has Jordan content zero (a set has Jordan content zero provided it is a subset of a closed set of Lebesgue measure zero). More generally: the function $f(z)=\sum a_n z^n$, holomorphic in $|z|<1$, has the unit circle for a natural boundary if there exist a sequence of pairs of integers (p_k, q_k) and an irrational number λ with the following properties:

(1) $p_k \to \infty$, $q_k - p_k > \theta p_k$, for some $\theta>0$,
(2) $\limsup |a_m|^{1/m}=1$,
(3) the set $\{\exp 2\pi i m\lambda\}$ has Jordan content zero;

in (2) and (3), the symbol m runs through all integers in $p_r \le m \le q_r$ $(r=1, 2, \cdots)$ for which $a_m \neq 0$.

G. Piranian (Ann Arbor, Mich.).

Dalovitch, V. Un critère que doit satisfaire une fonction analytique pour qu'elle appartienne à la classe H_δ $(1<\delta\le 2)$. Bull. Soc. Math. Phys. Serbie **6**, 80–85 (1954). (Serbo-Croatian. French summary)

The author proves for $1<\delta\le 2$ the following result [which is valid for all $\delta>1$, and is an obvious corollary of the well known result of M. Riesz concerning conjugate functions, Math. Z. **27**, 218–244 (1927)]: if $f(z)=u(z)+iv(z)$ is regular for $|z|<1$, $u(z)>0$, $v(0)=0$, and if $u^\delta(z)$ has a harmonic majorant, then f belongs to the class H_δ.

A. Zygmund (Chicago, Ill.).

★ **Rudin, Walter. Analytic functions of class H_p.** Lectures on functions of a complex variable, pp. 387–397. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A short account, without proofs, of the results obtained by the author in Trans. Amer. Math. Soc. **78**, 46–66 (1955) [MR **16**, 810].

W. W. Rogosinski (Newcastle-upon-Tyne).

Garabedian, P. R., and Schiffer, M. A proof of the Bieberbach conjecture for the fourth coefficient. J. Rational Mech. Anal. **4**, 427–465 (1955).

Let S be the family of functions

$$f(z)=z+a_2 z^2+\cdots$$

univalent in $|z|<1$. Bieberbach [S.-B. Preuss. Akad. Wiss. **1916**, 940–955] proved that, for $f(z) \in S$, $|a_2|\le 2$ holds, with equality only for the functions

$$f_\theta(z)=\frac{z}{(1-ze^{i\theta})^2}=z+2z^2e^{i\theta}+\zeta z^3 e^{2i\theta}+\cdots$$

and conjectured at the same time that for $\zeta \le n<\infty$, $|a_n|\le n$, with equality again only for the $f_\theta(z)$. This was proved for $n=3$ by Löwner [Math. Ann. **89** 103–121 (1923)]. The present authors prove the conjecture for $n=4$.

The paper consists of 3 chapters. In chapter I the differential equation satisfied by the extremal function which maximizes $|a_4|$ in S [M. Schiffer, Proc. London Math. Soc. (2) **44**, 432–449, 450–452 (1938)] is used to express a_n for $n>4$ in terms of a_2, a_3, a_4. Bieberbach (loc. cit.) noted already, that if $f(z) \in S$ and

$$[f(z^2)]^{\frac{1}{2}}=z+\frac{b_1}{z}+\frac{b_3}{z^3}+\cdots,$$

then

$$(*) \qquad \sum_{\nu=1}^{\infty}(2\nu-1)|b_{2\nu-1}|^2 \le 1.$$

The coefficients $b_{2\nu-1}$ here are expressible in terms of the a_n and hence in terms of a_2, a_3 and a_4.

In Chapter II the authors prove by taking up to 6 terms in (*), and heavy but mainly elementary calculations, that if $|a_4|\ge 4$, then one must have $|a_2|>1.989\cdots$, with a number of related results of similar accuracy.

Finally, in Chapter III Löwner's integral representation for the coefficients a_2, a_3 of the extremal function (loc. cit.) is used to show that a_2, a_3 satisfy equations deriving from a boundary-value problem of certain differential equations. It is then shown, that this problem has no solution for which a_2, a_3 approximate 2, 3 with the accuracy given in Chapter II, except that for which $|a_2|=2$, and hence $f(z)=f_\theta(z)$, $|a_4|=4$. Thus $|a_4|=4$ for the extremals $f_\theta(z)$, and $|a_4|<4$ for all other functions $f(z) \in S$.

The main reason for the length and difficulty of this work arises from the fact that the degree of approximation required for the results of Chapter III to go through is so close. In order to achieve it, formidable calculations are necessary in Chapter II. The method is nevertheless of the greatest interest. It not only carries Bieberbach's conjecture one step further from the point where Löwner placed it over 30 years ago, but also gives a hope, in principle at least, to prove the conjecture for the next one or two coefficients by an increase in labour, rather than an essentially new method.

W. K. Hayman (Exeter).

Charzyński, Zygmunt. Sur les fonctions univalentes algébriques bornées. Rozprawy Mat. **10**, 41 pp. (1955).

Given T between 0 and 1 and a natural number M, define the family $\mathfrak{A}_{TM}$ of functions of the form

$$a(z)=\alpha_1 z+\alpha_2 z^2+\cdots \quad (|z|<1),$$

which satisfy the two conditions: (1) $\alpha_1 \geq T$; (2) there exist two rational functions

$$\Omega(w)=\sum_{s=0}^{L}(D_s w^{-s}+\bar{D}_s w^s),\ \chi(z)=\sum_{s=0}^{L}(E_s z^{-s}+\bar{E}_s z^s),$$

where $D_L\neq 0$, $E_L\neq 0$, $1\leq L\leq M$, such that $\Omega(a(z))=\chi(z)$. If $\mathfrak{A}_T$ is the union of all the families $\mathfrak{A}_{TM}$ for all natural numbers M, then $\mathfrak{A}$, the union of all families $\mathfrak{A}_T$, is called by the author the family of bounded algebraic univalent functions. He proves that every bounded univalent function can be approximated arbitrarily closely by functions in $\mathfrak{A}$. Let $H(X_2, X_3, \cdots, X_N, Y_2, Y_3, \cdots, Y_N)$ be any real analytic function defined in a sufficiently large domain of the real variables X_i, Y_i $(i=2, 3, \cdots, N)$, and suppose that H has nonvanishing gradient. Define $H_a=H(\lambda_2, \lambda_3, \cdots, \lambda_N, \mu_2, \mu_3, \cdots, \mu_N)$, where $\lambda_n+i\mu_n=a_n$ $(n=2, 3, \cdots, N)$. The main theorem of this paper gives conditions satisfied by Ω and χ when a is an extremal function for which H_a attains its largest value in the family $\mathfrak{A}_{TM}$. *G. Springer* (Lawrence, Kansas).

Alenicyn, Yu. E. On univalent functions in multiply connected domains. Dokl. Akad. Nauk SSSR (N.S.) **102**, 861–863 (1955). (Russian)

A class of disjoint multiply connected regions $\{D_\nu\}$ is said to have a "completion" if it is possible to adjoin components of the complement of D_ν to D_ν such that the new regions obtained are simply connected, but still disjoint. Let B be a bounded, finitely connected z-region without isolated boundary points, and let a_ν, $\nu=1, 2, \cdots, 2m$, be the vertices of a regular polygon in the w-plane. Let $w=f_\nu(z)$, map B onto D_ν, $D_i\cap D_j=0$ $(i\neq j)$, $f_\nu(0)=a_\nu$, where it is assumed that $\{D_\nu\}$ has a completion. The author gives the best upper bound for $\prod_{\nu=1}^{2m}|f_\nu'(0)|$ in terms of B and $\{a_\nu\}$, generalizing results of Goluzin [Mat. Sb. N.S. **29(71)**, 455–468 (1951); MR **13**, 454], and others. Theorem 6 gives a best inequality for the Schwarzian derivative of a bounded function, schlicht in $\{|z|<1\}$. The author also discusses functions schlicht in the unit circle minus circular slits. The paper contains no proofs, but it is stated that the results are based on Nehari's work [Trans. Amer. Math. Soc. **75**, 256–286 (1953); MR **15**, 115]. *E. Reich* (Santa Monica, Calif.).

Reade, Maxwell O. On close-to-convex univalent functions. Michigan Math. J. **3**, 59–62 (1955).

Let $f(z)=\sum_0^\infty a_n z^n$, $\phi(z)=\sum_0^\infty c_n z^n$ and $g(z)=f'(z)/\phi'(z)=\sum_{n=0}^\infty g_n z^n$. The function $f(z)$ is called close-to-convex. [W. Kaplan, Michigan Math. J. **1**, 169–185 (1953); MR **14**, 966] if $f(z)$ is analytic in $|z|<1$ and $\phi(z)$ can be found giving a convex mapping of $|z|<1$, and such that $g(z)$ has positive real part. Kaplan showed that close-to-convex functions are univalent and that a large number of previously studied special classes of univalent functions are close-to-convex.

The present author now notes that if $f(z)$ is close-to-convex, then the well-known inequalities $|g_n|\leq 2|g_0|$, $|c_n|\leq|c_1|$ together with $na_n=nc_ng_0+\cdots+c_1g_{n-1}$, result in $|a_n|\leq n|a_1|$, which is Bieberbach's conjecture for close-to-convex functions. He also considers close-to-star functions, $f(z)=\sum_1^\infty a_n z^n$, for which $f(z)/\Psi(z)$ has positive real part, where $\Psi(z)$ is some mapping of $|z|<1$ onto a star-domain. In this case $f(z)$ is no longer univalent in general, and the sharp inequalities $|a_n|\leq n^2|a_1|$ hold.

W. K. Hayman (Exeter).

★ **Hayman, W. K. Uniformly normal families.** Lectures on functions of a complex variable, pp. 199–212. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A family of functions regular in $|z|<1$ is defined as uniformly normal and belongs to $\mathfrak{N}(\beta)$ defined for $\beta\geq 0$ if the conditions $|f(z_1)|\leq 1$ and $|f(z_2)|\geq B$ imply $|(z_1-z_2)/(1-\bar{z}_1z_2)|\geq\frac{1}{2}$. The author proves that if $f(z)=a_0+a_1z+a_2z^2+\cdots$ belongs to $\mathfrak{N}(\beta)$ then

$$|a_1|\leq 2\mu(\log\mu+\beta_1),$$

where $\mu=\max(1, |a_0|)$ and β_1 depends only on β. Also, $|f'(z)|\leq 2(1-r^2)^{-1}\mu^*(\log\mu^*+\beta_1)$ where $\mu^*=\max(1, |f(z)|)$ and $\log M(r)=O\{(1-r)^{-1}\}$ with $M(r)=\max|f(re^{i\theta})|$. The growth of the means

$$T_\lambda(r)=\frac{1}{2\pi}\int_0^{2\pi}\{\tfrac{1}{2}\log[1+|f(re^{i\theta})|^2]\}^\lambda d\theta$$

is also investigated.

A family of functions $w=\exp\{g(z)\}$ is shown to be uniformly normal if the schlicht circles $|w-iA|<R$ (where A is real) in the Riemann surface of each $g(z)$ are of uniformly bounded radius. It follows that such $g(z)$ satisfy the inequalities obtained by J. E. Littlewood [Proc. London Math. Soc. (2) **23**, 481–519 (1925)] in the case $g(z)\neq 2k\pi i$. *A. J. Macintyre* (Aberdeen).

Constantinescu, Corneliu. Au sujet du comportement des fonctions méromorphes au voisinage de la frontière. Com. Acad. R. P. Romîne **5**, 465–470 (1955). (Romanian. Russian and French summaries)

Contribution to the theory of the boundary behavior of functions meromorphic in $|z|<1$. Generalization of a uniqueness theorem of F. Wolf [Math. Ann. **117**, 383 (1940); MR **2**, 81]. *M. Heins* (Providence, R.I.).

★ **Shapiro, Harold S. Applications of normed linear spaces to function-theoretic extremal problems.** Lectures on functions of a complex variable, pp. 399–404. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A short outline of methods and results fully expounded in a joint paper by the reviewer and author [Acta Math. **90**, 287–318 (1953), MR **15**, 516].

W. W. Rogosinski (Newcastle-upon-Tyne).

Waadeland, Haakon. Über k-fach symmetrische, sternförmige schlichte Abbildungen des Einheitskreises. Math. Scand. **3**, 150–154 (1955).

Let S_k be the class of functions, $f(z)=z+\sum_2^\infty a_nz^n$, regular and schlicht for $|z|<1$, for which $f(ze^{2\pi i/k})=e^{2\pi i/k}f(z)$. Let S^*_k be the subclass of S_k whose elements map the unit circle onto a star-shaped region. Using two different methods, the author obtains precise bounds for a_n when $f\in S^*_k$. The result is $|a_n|=0$ when $n\not\equiv 1 \pmod k$,

$$|a_{pk+1}|\leq\prod_{\nu=1}^{p}\left(1-\frac{k-2}{\nu k}\right)\ (p=1, 2, \cdots).$$

Results of Fekete and Szegö [J. London Math. Soc. **8**, 85–89 (1933)] show that these bounds do not hold for S_k.

E. Reich (Santa Monica, Calif.).

Korickiĭ, G. V. On curvature of level lines and of their orthogonal trajectories in conformal mappings. Mat. Sb. N.S. **37(79)**, 103–116 (1955). (Russian)

Let K_r denote the curvature of the image curve of the circle $|z|=r$ under a conformal mapping, and similarly let K_{φ_0} denote the curvature of the image curve of the radial line $\arg z=\varphi_0$. The author finds sharp upper and lower bounds for K_r and K_{φ_0} for certain classes of univalent functions as indicated: (I) The class of functions of the form

$$F_p(z)=\int_0^z \prod_{m=1}^{n}(1-e^{i\pi\beta_m}z^p)^{-\mu_m}dz,$$

where p is a positive integer, $\sum_{m=1}^{n}\mu_m=2/p$, and $-1\leq\sum_{m=j}^{k}\mu_m\leq 1+2/p$ $(j,k,=1,2,\cdots,n)$. Here K_r is bounded. (II) The class of functions $z+\sum_{n=1}^{\infty}a_nz^{np+1}$ regular, univalent and convex in $|z|<1$. Here K_{φ_0} is bounded. (III) The class of functions $\zeta+\sum_{n=1}^{\infty}p_n\zeta^{1-np}$ regular and univalent in $1<|\zeta|<\infty$ and mapping $|\zeta|>1$ onto a domain with convex complement. Here K_r and K_{φ_0} are bounded.

The formulas giving these sharp bounds are too complicated to be reproduced here.

A. W. Goodman (Lexington, Ky.).

Pesin, I. N. On the theory of generalized Q-quasiconformal mappings. Dokl. Akad. Nauk SSSR (N.S.) **102**, 223–224 (1955). (Russian)

A family $C(z, k)$ of curves C_α $(0<\alpha<1)$ is called regular at z (with parameter k) if there exists a homeomorphic mapping of the disk $|t|<1$ onto some neighborhood of z which maps $t=0$ onto the point z and each circle $|t|=\alpha$ onto the corresponding curve C_α, and if

$$\limsup_{\alpha\to 0}\max_{z'\in C_\alpha}|z'-z|/\min_{z'\in C_\alpha}|z'-z|=k.$$

A homeomorphic mapping (1) $w=f(z)$ of $|z|<1$ onto $|w|<1$ is called regular at z if it transforms some regular family $C(z, k')$ into a regular family $C(f(z), k'')$. The lower bound $q(z)=\inf k'k''$, taken with respect to all regular curve families $C(z, k')$, is the characteristic of (1). And (1) is a generalized Q-quasiconformal mapping if it is regular at each point in $|z|<1$ and if vrai max $q(z)<Q$. This definition should be compared with a definition suggested by Lavrentieff [Mat. Sb. **42**, 407–423 (1935); see p. 423].

The author shows that for every generalized Q-quasiconformal mapping the set of points of differentiability with zero Jacobian has measure zero, and that each sequence of generalized Q-quasiconformal mappings has a subsequence converging uniformly to a generalized Q-quasiconformal mapping.

G. Piranian (Ann. Arbor, Mich.).

Belinskiĭ, P. P., and Pesin, I. N. On the closure of a class of continuously differentiable quasi-conformal mappings. Dokl. Akad. Nauk SSSR (N.S.) **102**, 865–866 (1955). (Russian)

Let $p(z)$ and $\theta(z)$ be measurable characteristics $(p(z)<Q)$, defined for almost all z $(|z|<1)$; then there exists a unique function $w=f(z)$ for $f(0)=0$ and $f(1)=1$, which effects a generalized Q-quasiconformal mapping (see the preceding review) of the disc $|z|\leq 1$ onto the disc $|w|\leq 1$, and which possesses almost everywhere the given characteristics. From the authors' proof of this theorem it follows that the class of generalized Q-quasiconformal mappings is the closure of the class of quasiconformal mappings with continuous characteristics.

G. Piranian (Ann Arbor, Mich.).

Kolbina, L. I. Conformal mapping of the unit circle onto mutually nonoverlapping regions. Vestnik Leningrad. Univ. **10**, no. 5, 37–43 (1955). (Russian)

Let a_1, a_2, a_3, be fixed complex numbers and let $f_k(\zeta)$ map $|\zeta|<1$ conformally onto a simple region D_k containing $f_k(0)=a_k$ $(k=1, 2, 3)$, where $D_j\cap D_k$ is empty for $j\neq k$. The author seeks to maximize the quantity

$$I=\prod_{k=1}^{3}|f_k'(0)|^{\alpha_k}$$

for fixed $\alpha_k>0$. Variational methods lead to a differential equation from which she deduces

$$I_0\leq A(\alpha_1,\alpha_2,\alpha_3)|a_1-a_2|^{\alpha_1+\alpha_2-\alpha_3}|a_2-a_3|^{-\alpha_1+\alpha_2+\alpha_3}|a_1-a_3|^{\alpha_1-\alpha_2+\alpha_3}.$$

However, details of the integration have been omitted and the explicit expression for the multiplier $A(\alpha_1, \alpha_2, \alpha_3)$ is not given. A similar problem with four regions is completely solved, in the special case that $a_1=a$, $a_2=-a$, $a_3=0$, $a_4=\infty$, and $\alpha_1=\alpha_2=\alpha_3=-\alpha_4=1$.

A. W. Goodman (Lexington, Ky.).

Albrecht, Rudolf. Iterationsverfahren zur konformen Abbildung eines Ringgebietes auf einen konzentrischen Kreisring. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 169–178 (1955).

A classical theorem concerning conformal mapping of a doubly-connected domain onto an annulus is proved by means of an iteration process analogous to those given by E Graeser [Dissertation, Leipzig, 1930] and by the reviewer [Proc. Jap. Acad. **21**, 146–155 (1949); MR **11**, 341]. Special attention is given to practical approximation for the mapping function.

Y. Komatu (Tokyo).

Kusunoki, Yukio. Note on the continuation of harmonic and analytic functions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29**, 11–16 (1955).

The problem of the removability of a closed set E of a Riemann surface R for the classes HB, HD, AB, AD (relative to $R-E$) is studied. The principal results pertain to Riemann surfaces which are given as covering surfaces of $|z|<1$. It is shown that, if the surface is given as an m-sheeted covering surface of $|z|<1$ and the product of the moduli of the ramification points is divergent, and if $m-1$ simple disks are given with mutually disjoint projections, then at least one of the disks is removable for AB and AD (relative to the surface less the disks). Use is made of the argument of P. J. Myrberg. For the case of a Riemann surface given as a covering surface of $|z|<1$ which has only algebraic branch points and has the property that the product of the moduli of the ramification points converges, it is shown that every closed domain E of the surface is not removable for AB (relative to $R-E$). Use is made of the special properties of the surfaces considered.

M. Heins (Providence, R.I.)

Kuramochi, Zenjiro. On the behaviour of analytic functions on abstract Riemann surfaces. Osaka Math. J. **7**, 109–127 (1955).

It is known that classes O_F of Riemann surfaces characterized by the nonexistence of suitably bounded harmonic functions F are strictly larger than the class O_G of parabolic Riemann surfaces. These recent results have opened the way to the problem of determining what properties holding in O_G remain valid in a larger class O_F. The author gives the following contribution: the Gross

property, known to hold in O_G, is not shared by all Riemann surfaces in O_{HP}.

The author also proves that if $W \notin O_G$ but $W \in O_{HB}$, then the complement of any compact $W_0 \subset W$ belongs to O_{AB}. He also discusses the Iversen property for classes related to O_{AB}. *L. Sario* (Los Angeles, Calif.).

Kuramochi, Zenjiro, and Kuroda, Tadashi. A note on the set of logarithmic capacity zero. Proc. Japan Acad. **30**, 566–569 (1954).

Let G be a Riemann surface imbedded in a Riemann surface P and let the relative boundary of G consist of analytic curves C. Let G_1 be obtained from G by symmetrization with respect to C. Let a function $u(p)$ exist which is harmonic on G, continuous on $G \cup C$, and approaching $+\infty$ on the ideal boundary of G. Let the conjugate $v(p)$ of $u(p)$ have uniformly bounded variation $\int dv$ on the level curves $u(p)=r$. It is then proved that G_1 has null boundary in the sense of Nevanlinna [Uniformisierung, Springer, Berlin, 1953, p. 319; MR **15**, 208]. This is applied to obtain a necessary and sufficient condition that a closed subset of $|z|=1$ have capacity zero. *W. Kaplan* (Ann Arbor, Mich.).

Kuramochi, Zenjiro. On the existence of harmonic functions on Riemann surfaces. Osaka Math. J. **7**, 23–28 (1955).

The author gives a new proof of a theorem of the reviewer [Proc. Amer. Math. Soc. **5**, 266–269 (1954); MR **15**, 695] which states that the existence of an HD function on an open Riemann surface is a property that is preserved under quasi-conformal mapping. The present proof makes extensive use of the universal covering surface of the original surface.

H. L. Royden (Stanford, Calif.).

Galbură, G. Sur le genre d'une courbe algébrique. Com. Acad. R. P. Române **3**, 105–107 (1953). (Romanian. Russian and French summaries)

The Riemann theorem relating the genus, the degree and the ramification index of an algebraic function is established with the aid of the index relations of the theory of pseudo-harmonic functions.

M. Heins (Providence, R.I.)

Tietz, H. Berichtigung der Arbeit „Zur Realisierung Riemannscher Flächen." Math. Ann. **129**, 450 (1955). See Math. Ann. **128**, 453–458 (1955); MR **16**, 688.

Fréchet, Maurice. Les surfaces dérivables relativement à une règle de multiplication (en deux mémoires). Mémoire préliminaire: Sur certains systèmes d'équations aux dérivées partielles et sur certaines familles de surfaces. Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Reeks I. **21**, no. 1, 44 pp. (1954).

The author's results are published in two memoirs, of which the second is reviewed below. The latter is a theory of n-dimensional "para-analytic" functions in the author's sense [C. R. Acad. Sci. Paris **236**, 1832–1834; MR **15**, 117], while in the present paper the author studies a certain system t of partial "differential equations" (subject to his multiplication law R [see the following review])

$$\sum_{r=1}^{p}\sum_{k=1}^{n} C_{krh}\frac{\partial X_k}{\partial x_r}=0 \quad (h=1, \cdots, q)$$

(with constant coefficients) under the "coordinate transformations"

$$X_h=\sum_{i=1}^{n}\lambda_{ih}Y_i+p_h \quad (h=1, \cdots, n),$$

$$y_m=\sum_{r=1}^{p}\mu_{mr}x_r+\nu_m \quad (m=1, \cdots, p)$$

and their geometrical interpretation.

The first memoir consists of two parts: analytical exposition and geometrical interpretation. The system $\sigma: n=3$, $p=2$, $q=3$ is treated concretely in detail. The complete system of "canonical forms" of the system σ is tabulated in Table II (p. 21) (Table II in the second memoir, p. 39) and the different forms to which the solutions of the system σ are reducible are tabulated in Table I (which is Table I in the second memoir, p. 38). The "canonical forms" of the partial differential equations" of the third order with constant coefficients, which the three X_h satisfy, are tabulated in Table III (Table IV in the second memoir, p. 51). The confluences of one or more of the families of this table under the "coordinate transformations" are also studied; it is shown that confluence is possible. The second part consists of a detailed study of the so-called "derivable surfaces" (subject to the multiplication law R) $X(u, v)$, $Y(u, v)$, $Z(u, v)$. Equivalences $X(u, v)=\xi(u_1, v_1)$, etc., under the parameter transformations $u=u(u_1, v_1)$, $v=v(u_1, v_1)$ are studied. The "canonical equations" of the "derivable surfaces" are discussed and tabulated in Table V (Table VII in the second memoir, p. 56). Equivalence of the equations of one and the same "derivable surface" is studied, and it is shown that this is possible. The "canonical functions" $\phi(X, Y)$ in the "derivable surfaces" (with respect to R) as solutions of a linear partial "derived equations" of the third order with polynomial coefficients and the confluences of the family of the "derivable surfaces" (parallel to the confluences spoken of above) are also discussed.

T. Takasu (Yokohama).

Fréchet, Maurice. Les surfaces dérivables relativement à une règle de multiplication. Ann. Sci. Ecole Norm. Sup. (3) **71**, 29–85 (1954).

This paper is a detailed exposition of an earlier note [C. R. Acad. Sci. Paris **237**, 1053–1055 (1953); MR **15**, 521], which is a generalisation of a series of the same author's papers [cf. MR **15**, 117, 416, 521, 697]. It consists of Part I (analytic exposition) and Part II (geometrical interpretation).

Part I. $e_1, \cdots, e_p$ are unit vectors on p axes; $f_1, \cdots, f_n$ are unit vectors on n axes. A function $W=\Phi(w)=X_1f_1+\cdots+X_nf_n$ of $w=x_1e_1+\cdots+x_pe_p$ is considered, where $X_i=X_i(x_1, \cdots, x_p)$. Derivability is distinguished from differentiability. In conformity with the author's general definition [Ann. Sci. Ecole Norm. Sup. (3) **42**, 293–323 (1925), p. 297–301] of differentiability in a Banach-Hahn space, $\Phi(w)$ is said to be differentiable for $w=w_0=\sum_r x_{r_0}e_r$ if there exists a linear functional $L(\Delta w)$ of Δw (in the same space as that of $\Phi(w)$ in the sense that $L(\Delta w)=\sum_k L_k(\Delta w)f_k$, where $L_k(\Delta w)$ is a linear functional of Δw) such that $\Delta\Phi(w)-L(\Delta w)=\|\Delta w\|\varepsilon$, where $\|\Delta w\|$ is a norm of Δw (e.g., $\|\Delta w\|=(\sum_r(\Delta x_r)^2)^{\frac{1}{2}}$ or $\|\Delta w\|=\sum_r|\Delta x_r|$, etc.) and $\|\varepsilon\|\to 0$ ($\|\Delta w\|\to 0$); then $L(\Delta w)$ is called the differential $d\Phi$ of $\Phi(w)$ in $w=w_0$. It follows that $\Delta\Phi(w)-L(\Delta w)$ is of the form $\sum[\Delta X_k-L_k(\Delta w)]f_k$, so that $\varepsilon=\sum\varepsilon_kf_k$ with $\Delta X_k-L_k(\Delta w)=\varepsilon_k\|\Delta w\|$, where $\lim_{\|\Delta w\|\to 0}\varepsilon_k=0$ and L_k

is a linear functional of Δw. This implies: in order that $\Phi(w)=\sum X_k f_k$ may be differentiable in $w=w_0$, it is necessary and sufficient that each $X_k(x_1, \cdots, x_n)$ be differentiable in $w=w_0$ in the sense of Stolz-Young, and then $d\Phi=\sum_k dX_k f_k$.

A multiplication rule, according to which we have

$$(\sum_k \xi_k f_k)(\sum_r x_r e_r)=\sum_k \sum_r \xi_k x_r (f_k \cdot e_k)$$

with the rule R: $f_k \cdot e_r=\sum_k a_{krk} f_k$ (a_{krk} being real constants arbitrarily chosen), is introduced and it is said that the hypercomplex function

$$\Phi(w)=X_1(x_1, \cdots, x_n)f_1+\cdots+X_n(x_1, \cdots, x_n)f_n$$

of the hypercomplex variable $w=x_1e_1+\cdots+x_ne_n$ is "derivable" at $w=w_0=x_{10}e_1+\cdots+x_{n0}e_n$ subject to the multiplication rule R, if 1) $d\Phi=\sum dX_k f_k$ is differentiable at $w=w_0$; 2) there exists only one hypercomplex number $\Phi'(w_0)=\sum \Phi'_k(w_0)f_k$ independent of the direction dw such that $d\Phi(w)=\Phi'(w_0)dw$ in $w=w_0$, the multiplication being subject to the rule R. Then $\Phi'(w_0)$ is called the "derivative" of $\Phi(w)$ in $w=w_0$ subject to the rule R. It is said that $\Phi(w)=\sum_k X_k f_k$ is a para-analytic function of w at $w=w_0$ subject to the rule R. Change of variables and change of complex units are explained. Analogues of the Cauchy-Riemann equations for a hypercomplex function $\Phi(w)=\sum X_k f_k$ subject to R in the vicinity of w_0 are developed. Canonical forms of derivable functions (subject to R) are given. Necessary and sufficient conditions that $\Phi(w)$ be para-analytic subject to the rule R are given. Confluent families of derivable functions are also given.

Part II. Geometrical interpretations similar to the case $X=X(u, v)$, $Y=Y(u, v)$, $Z=Z(u, v)$ of ordinary surface theory are given in detail, and so-called canonical forms are especially considered. [The following papers should be consulted: T. Takasu, Yokohama Math. J. **1**, 131–224 (1953); MR **16**, 350; and the paper reviewed second below].

T. Takasu (Yokohama).

Fréchet, Maurice. The canonical form of the 2, 3, and 4-dimensional para-analytic function. Compositio Math. **12**, 81–96 (1954). (Esperanto. French summary)

The author summarizes first the definitions and the properties of the 'derivable' (distinguished from 'differentiable') n-dimensional functions and the para-analytic n-dimensional functions subject to his law R_p of multiplication [cf. the preceding review], as given in his résumé [C. R. Acad. Sci. Paris **237**, 1053–1055 (1953); MR **15**, 521] and in the detailed expositions [ibid. **236**, 2364–2366 (1953); Ann. Mat. Pura Appl. (4) **35**, 255–268 (1953); MR **15**, 416, 697]. He then shows that in case $n=2$, 3 and 4 all these functions may be brought, by simples processes, to their canonical forms, which he tabulates, giving 'Cauchy-Riemann' equations, 'Laplacians' etc. The last table for the case $n=4$, containing 130 formulas for five canonical functions, will suffice to suggest how the cases $n>4$ become and what kind of special problems may arise. [The following papers should be compared: Takasu, the paper reviewed below and Yokohama Math. J. **1**, 131–224 (1953); Proc. Phys.-Math. Soc. Japan (3) **25**, 209–220 (1943); Sci. Rep. Tôhoku Imp. Univ. Ser. 1. **32**, 1–55 (1945); MR **16**, 350; **7**, 291; **10**, 445; Yoneda, Yokohama Math. J. **1**, 225–262 (1953); MR **16**, 351; Nisigaki, reviewed second below.

T. Takasu (Yokohama).

Takasu, Tsurusaburo. A complex function theory on a "supra-corpus" of n-dimensional hypercomplex numbers. II. A theory of triply periodic functions. Yokohama Math. J. **2**, 1–68 (1954).

[For part I see same J. **1**, 131–224 (1953); MR **16**, 350.] Let $f(\zeta)$ be a function of the n-dimensional hypercomplex function ζ. It is periodic if there exists a hypercomplex number ω not a nilfactor such that $f(\zeta+\omega)=f(\zeta)$ for all ζ. After developing the elementary properties of simply periodic functions, the author develops in parallel columns the theory of doubly and triply periodic functions. This development includes theorems analogous to those on elliptic functions given by Liouville and Weierstrass.

J. A. Ward (Holloman, N.M.).

Nisigaki, Hisami. A theory of functions of a ternary complex variable. I. Mem. School Sci. Engrg. Waseda Univ., Tokyo no. **18**, 110–122 (1954).

For any linear, commutative, associative algebra of order three with an identity over the real field it is shown how to choose a basis I, E_2, E_3 such that I is the identity, $E_2^2=\alpha E_2+\beta E_3$, $E_3^2=\gamma E_2+\delta E_3$, $E_2E_3=\varepsilon I+\delta E_2+\alpha E_3$ with $\varepsilon=\beta\gamma-\alpha\delta$. Then E_2 and E_3 satisfy the respective scalar equations

$$x^3-2\alpha x^2+(\delta^2-\beta\delta)x-\beta\varepsilon=0, \quad y^3-2\delta y^2+(\delta^2-\alpha\gamma)y-\gamma\varepsilon=0.$$

If $\beta\gamma\neq 0$, the author shows that $y=(\delta x+\varepsilon)/(x-\alpha)$ and that, if the first cubic equation has distinct roots, exactly two equal roots, or three equal roots, then so does the second. In each of these he shows how to choose a basis I_1, I_2, I_3 such that $I=I_1+I_2+I_3$ and each I_i is idempotent or nilpotent. In terms of the conjugate roots of the cubics, conjugates to E_2 and E_3 are determined. For $X=x_1I+x_2E_2+x_3E_3$ the author defines $X'=x_1I+x_2E_2'+x_3E_3'$ and $X''=x_1+x_2E_2''+x_3E_3''$.

He defines $N(X)=XX'X''$ to be the norm of X. Then $N(X)$ has the properties $N(X)=N(X')=N(X'')$, $N(XY)=N(X)N(Y)$ and $N(X)=0$ if and only if X is nilpotent. The manifold $N(X)=0$ is called the nilfactor of the algebra.

J. A. Ward (Holloman, N.M.).

See also Bernšteĭn, p. 31

Harmonic Functions, Potential Theory

Huber, Alfred. The reflection principle for polyharmonic functions. Pacific J. Math. **5**, 433–439 (1955).

The author shows that if $w(x_1, \cdots, x_n)$ satisfies the differential equation $\Delta^p w=0$ in a region G whose boundary contains an open subset S of $x_1=0$ and if w/x_1^{p-1} assumes the boundary value 0 on S, then w may be continued analytically across S into the reflection of G.

H. L. Royden (Stanford, Calif.).

Magenes, Enrico. Problema generalizzato di Dirichlet e teoria del potenziale. Rend. Sem. Mat. Univ. Padova **24**, 220–229 (1955).

Let D denote a domain in space and use FD to indicate its boundary. If the boundary FD is sufficiently smooth and the assigned boundary values $\mu=\mu(p)$ are summable, then a potential of a double layer can be set up in D whose limiting values (along normals to FD) take on the values μ almost everywhere on FD [G. C. Evans and E. R. C. Miles, Amer. J. Math. **53**, 493–516 (1931)]. Another variation of the Dirichlet problem is to assume the boundary values μ are taken on in the mean of order p [G. Cimmino, Rend. Circ. Mat. Palermo **61**, 177–221

(1937); Rend. Sem. Mat. Fis Milano, **23**, 183–203 (1953); MR **16**, 133]. The author shows that the above generalizations of the Dirichlet problem lead to the same class of solutions if the boundary values μ belong to L^p $(p>1)$. The possibility of treating similar boundary-value problems for more general elliptic equations and also their extensions to parabolic equations is discussed at length in the last part of this article.

F. G. Dressel (Durham, N.C.).

See also: Kuramochi, p. 27; Bernštein, p. 31.

Series, Summability

★**Gloden, A. Le développement des procédés de sommation des séries divergentes.** Actes du VI^{e} Congrès International d'Histoire des Sciences, Amsterdam, 1950, pp. 178–186. Hermann & Cie, Paris, 1951.

★**Teghem, J. Sommation de séries divergentes et prolongement analytique.** III^{e} Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 20–23. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author outlines some of the results obtained by himself and others on generalized Euler summability of series. *R. C. Buck* (Madison, Wis.).

Davydov, N. A. A generalization of Abel's second theorem. Uspehi Mat. Nauk (N.S.) **10**, no. 3(65), 135–138 (1955). (Russian)

Assuming that s_{n_k-1} converges, the author proves A-summability of $s_n=\sum_1^n a_k$ under some additional assumptions, such as $\sum(1-n_k/p_k)\varrho_k<+\infty$, where $\varrho_k=\sum_{n_k}^{p_k}|a_n|$ and $a_n=0$ for $p_k<n<n_{k+1}$. [The reviewer remarks that stronger results with C_1 instead of A are obtained by replacing the s_n by properly chosen s_{n_k} in the expression of the C_1-means.] He also gives results which imply that the Hardy-Littlewood "high-indices" theorem cannot be improved. [A more general theorem is found in G. Lorentz, Trans. Amer. Math. Soc. **63**, 226–234 (1948); MR **9**, 425.]

G. G. Lorentz (Detroit, Mich.).

van Dantzig, D. Sur un problème de M. Karamata. Nieuw Arch. Wisk. (3) **3**, 89–92 (1955).

L is said to be a strong limit point for a sequence $\{a_n\}$ if there is a subsequence $\{a_{n_k}\}$ which converges to L, with $n_{k+1}/n_k=O(1)$. Let $\{a_n\}$ be a real sequence such that $\sum n^{-1}a_n$ has bounded partial sums, while $a_{n+1}-a_n=O(n^{-1})$. The author answers a question raised by Karamata by showing that it is not necessary that $\{a_n\}$ have 0 as a strong limit point. This he does by exhibiting a continuous function f such that: 1) $|f(x)-f(y)|\leq|x-y|$ for all $x>0$ $y>0$; 2) $|\int_0^x f|\leq 1$ for all $x>0$, but such that there does not exist a sequence $\{x_n\}$ with $x_n\uparrow\infty$, $x_{n+1}-x_n=O(1)$, and $\lim f(x_n)=0$. *R. C. Buck* (Madison, Wis.).

Macintyre, Sheila Scott. On a problem of Ramanujan. J. London Math. Soc. **30**, 310–314 (1955).

Let

$$R(n)=\frac{1}{n+1}+\sum_2^\infty\frac{k^{k-2}}{(n+k)^k}.$$

Ramanujan posed the problem: show that $R(n)<1/n$, and in particular, show that for $n=1000$,

$$\Delta(n)=1/n-R(n)\sim(1.0125)10^{-440}$$

[J. Indian Math. Soc. **6**, 39 (1914)]. G. Szegö showed that $\Delta(n)<2e^{-n}/n(n+2)$ which in particular gives the estimate $\Delta(1000)\sim(1.0132)10^{-440}$ [J. London Math. Soc. **3**, 225–232 (1928)]. The author obtains a more precise asymptotic estimate for $\Delta(n)$, which in a special case gives

$$\Delta(n)=e^{-n}\left\{\frac{2n+1}{n(n+1)^2}-\frac{7}{3(n+1)^3}+\frac{17}{3(n+1)^4}+O(1/n^5)\right\}$$

justifying the value for $\Delta(1000)$ given by Ramanujan. Let $F(z)=1/(n+z+1)$ and consider the Abel series associated with F, $S(z)=F(0)+\sum_1^\infty F^{(k)}(k)z(z-k)^{k-1}/k!$. The author notes that $F(-1)=1/n$ while $S(-1)=R(n)$ so that $\Delta(n)=F(-1)-S(-1)$. She then estimates this by means of recent results in a paper of hers and A. J. Macintyre on the convergence and asymptotic behavior of Abel series [Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 222–231 (1952); MR **13**, 738]. *R. C. Buck* (Madison, Wis.).

Rebollo, J. M. Determination of the scale of recurrence of a series. Rev. Acad. Ci. Madrid **48**, 203–215 (1954). (Spanish)

It is well known that if the coefficients of a Taylor series $\sum_{n=0}^\infty c(n)z^n$ satisfy a recursion relation of the form (*) $\sum_{k=0}^m b(k)c(n-k)=0$ for some fixed integer m and all $n>p$ then $\sum_{n=0}^\infty c(n)z^n$ defines a rational function whose denominator is the polynomial $\sum_{n=0}^m b(n)z^n$. The author obtains conditions on the coefficients $c(n)$ in order that they satisfy a relation of the type (*) above. For example, if $c(n+h)=a_{1h}f_1(n)+a_{2h}f_2(n)+\cdots+a_{rh}f_r(n)$, then in general the $c(n)$ will satisfy such a recursion relation. Using this fact, the author expresses a number of Taylor series in closed form.

V. F. Cowling (Lexington, Ky.).

Erdélyi, A. Asymptotic representations of Fourier integrals and the method of stationary phase. J. Soc. Indust. Appl. Math. **3**, 17–27 (1955).

This largely expository paper is a rapid survey of results (along with some historical remarks) about the asymptotic behavior of (*) $\int_a^b g(t)e^{ixh(t)}dt$ for large real x when $g(t)$ and $h(t)$ are assumed to be continuously differentiable except possibly at a finite number of points.

The last of the four theorems describes the behavior of (*) when $g(t)=(t-\alpha)^{\lambda-1}(\beta-t)^{\mu-1}g_1(t)$ where $0<\lambda$, $\mu\leq 1$, and $h'(t)=(t-\alpha)^{\varrho-1}(\beta-t)^{\sigma-1}h_1(t)$ where ϱ, $\sigma\geq 1$, and $g_1(t)$, $h_1(t)$ are N times continuously differentiable for $\alpha\leq t\leq\beta$. The proof of the theorem depends on the idea of a neutralizer, a change of variable of integration, successive integration by parts, the Riemann-Lebesgue lemma and the method of steepest descents.

T. E. Hull (Vancouver, B.C.).

Farinha, João. Une condition de convergence uniforme. Rev. Fac. Ci. Univ. Coimbra **23**, 17–20 (1954).

The following theorem is proved. Let E denote a bounded closed set in the complex plane which does not contain 0. If the even and odd parts of the continued fraction $f=1/1+a_1/1+a_2/1+\cdots$ converge uniformly for a_i in E, then f converges uniformly for a_i in E. [The reviewer, in proving this, obtained the following more general theorem. If $\{a_i\}_{i=1}^\infty$ is a sequence of complex numbers having a bounded infinite subsequence and the even and odd parts of the continued fraction f converge, then f converges.] *H. S. Wall* (Austin, Texas).

See also: Ricci, p. 24.

Interpolation, Approximation, Orthogonal Functions

★ **Tricomi, Francesco G. Vorlesungen über Orthogonalreihen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXVI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+264 pp. DM 37.60.

The book is essentially a translation of an earlier work of the author published in Italian [Serie ortogonali di funzioni, Gheroni, Torino, 1948; MR **10**, 188]. Several new features have been added; this review will point out only a few details. 1) In Ch. 1, the conditions of Lauricella, Vitali, and Dalzell for the completeness of a system of functions deserve interest; they are not too well known. 2) The property of the partial sums of the Fourier series discussed in Ch. 2, p. 75 (cf. also p. 124, footnote) is of course due to L. Fejér [Math. Ann. **64**, 273–288 (1907), p. 281; M. Picone, 1924, is quoted]. 3) Ch. 5 deals with various useful properties of the hypergeometric and confluent hypergeometric functions complementing the material contained in the (somewhat "obsolete") book of the reviewer [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; MR **1**, 14]. 4) In Ch. 5 (orthogonal polynomials in finite intervals) more references seem desirable, for instance on p. 176 (zeros of Jacobi polynomials). 5) This is better in Ch. 6 (orthogonal polynomials in infinite intervals). The important results of the author on asymptotic formulas and zeros of Laguerre polynomials are only sketched (§ 2); it is only briefly mentioned that the proofs are based on the "method of Fubini". The equiconvergence theorem on Laguerre series due to the reviewer (§ 6) is only formulated. A more detailed treatment is given to the equiconvergence theorem for Jacobi series (§ 4); in fact this subject belongs to Ch. 5 since the interval is finite.

All in all the book is a fine didactic accomplishment.

G. Szegő (Stanford, Calif.).

Lo Ho. A new theory of interpolation for function of n variables. Acta Math. Sinica **4**, 125–142 (1954). (Chinese. English summary)

This paper describes the construction of a new function $\sum_{i=1}^{r} H_n(t, u, v, \cdots, z)_i$ of n variables $t, u, v, \cdots, z$, which becomes identical to the given function $f(t, u, v, \cdots, z)$ of n variables when $(n-1)$ of the variables are each assigned their own specific values used in the construction of the function. In case of two variables, the surfaces $f(t, u)$ of and $\sum_{i=1}^{r} H_2(t, u)_i$ coincide along $4r$ sections at the values

$$t_1,\ t_1',\ t_2,\ t_2',\ \cdots,\ t_r,\ t_r',$$
$$u_1,\ u_1',\ u_2,\ u_2',\ \cdots,\ u_r,\ u_r'.$$

If both are continuous and single-valued, actual examples indicate that the new function $\sum_{i=1}^{r} H_2(t, u)_i$ is a very close approximation of the original function $f(t, u)$ within the region of cross-lines of coincidence. *Author's summary.*

Berman, D. L. Investigation of the convergence of interpolation processes. Dokl. Akad. Nauk SSSR (N.S.) **102**, 867–869 (1955). (Russian)

Let $l_k^{(n)}(x)$ be the kth Lagrange interpolation polynomial for the points $x_k = x_k^{(n)}$ $(k=1, 2, \cdots, n)$. Put

$$U_n(f) = \sum f(x_{k-\eta}) l_k^{(n)}(x) \quad (\eta=0 \text{ or } 1 \text{ as } k \text{ is odd or even}),$$

$$V_n(f) = \tfrac{1}{4} \sum_{k=1}^{n} (f(x_{k-1}) + 2f(x_k) + f(x_{k+1})) l_k^{(n)}(x) \qquad (x_{n+1}=x_n,\ x_0=x_1),$$

$$W_n(f; x) = \tfrac{1}{16} \sum_{k=1}^{n} \sum_{j=-2}^{2} \binom{4}{2+j} f(x_{k+j}) l_k^{(n)}(x) \qquad (x_{-1}=x_0=x_1;\ x_{n+2}=x_{n+1}=x_n).$$

If the points $x_k^{(n)}$ are evenly spaced between $x_1^{(n)}=-1$ and $x_n^{(n)}=1$, then for $f(x)=x$ $\lim_{n\to\infty} V_n(f)=x$ $(|x|\leq 1)$, $\lim_{n\to\infty} U_n(f)$ does not exist in $|x|<1$, $x\neq 0$. This remains true if in the definition of $U_n(f)$ the $f(x_k^{(n)})$ are replaced by $(2/n)\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t)dt$. If $f(x)\equiv|x|$, $\lim_{n\to\infty} W_n(f)$ does not exist in $(-1, 1)$. If $x_k^{(n)}=-\cos((2k-1)/n\pi)$ and $f\in C(-1, 1)$, $|W_n(f; x)-f(x)|\leq(12\sqrt{2}+2.5|T_n(x)|\pi^2)\omega(\pi/n)$. ($\omega$ modulus of continuity, T_n Chebyshev polynomial). This gives a new proof of Jackson's theorem that the best approximation $E_n(f)$ by polynomials of degree n satisfies $E_n(f)=O(\omega(1/n))$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Eweida, M. T. On Newton's series of interpolation. Math. Z. **62**, 352–353 (1955).

Let $N(z)$ be Newton's interpolation series

$$f(a_1)+(z-a_1)f(a_1, a_2)+(z-a_1)(z-a_2)f(a_1, a_2, a_3)+\cdots,$$

where $f(a_1, a_2)$, $f(a_1, a_2, a_3)$, $\cdots$ are the divided differences of $f(z)$. If (i) $\sup|a_n|=A<\infty$, (ii) $\sum_{k=1}^{n}|a_k|=o(n)$, (iii) $f(z)$ is regular in $|z|<R$, $R>A$, then $N(z)$ converges uniformly in $|z|\leq R_1<R$. Examples are given to show that (i) and (ii) are necessary. *W. H. J. Fuchs.*

Gel'fond, A. O. On uniform approximations by polynomials with integral rational coefficients. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 41–65 (1955). (Russian)

Dans l'introduction sont exposés les principaux résultats concernant la possibilité d'approcher les fonctions définies sur un segment par des polynômes à coefficients entiers: l'approximation uniforme d'une fonction f continue sur un segment de longueur l est toujours possible si $l<1$, impossible si $l>4$ (à part les cas triviaux), et elle nécessite certaines conditions arithmétiques si $1<l<4$ [pour des conditions nécessaires et suffisantes, cf. Fekete, C. R. Acad. Sci. Paris **239**, 1337–1339, 1455–1457 (1954); MR **16**, 694]; l'approximation dans L_λ $(\lambda>0)$ est possible, sans conditions arithmétiques, si $l<4$ [pour $\lambda=2$, l'auteur cite E. Aparisio, Dissertation, Moscow, 1954]. Les paragraphes suivants sont consacrés à des théorèmes liant la rapidité de l'approximation uniforme sur le segment [0, 1] aux propriétés locales de la fonction f approchie: divisabilité d'ordre m, analyticité, etc.; moyennant les hypothèses naturelles sur f aux points 0 et 1, l'auteur montre que les énoncés de Jackson et S. Bernstein (relatifs à l'approximation par des polynômes quelconques) valent encore, quand il s'agit d'approximation par des polynômes à coefficients entiers.

J. P. Kahane (Montpellier).

Freud, Géza. Über einseitige Approximation durch Polynome. I. Acta Sci. Math. Szeged **16**, 12–28 (1955).

The following theorem on approximations is proved. Let $f(x)$, $-1\leq x\leq 1$, be differentiable $(v-1)$-times and $f^{(v-1)}(x)$ the integral of a function of bounded variation with variation V. Then we can find two polynomials $p(x)$, $P(x)$ of degree N such that $p(x)\leq f(x)\leq P(x)$;

$$\int_{-1}^{1} [P(x)-p(x)](1-x^2)^{-\frac{1}{2}}dx < A_v \cdot V \cdot N^{-v-1}.$$

Here A_v depends only on v. This result is used in order to estimate the error (1) in certain mechanical-quadrature formulas, (2) in a Tauberian formula for Laplace-Stieltjes transforms. *G. Szegő* (Stanford, Calif.).

Kaluza, Theodor, Jr. Mittlere Approximation mit linear unabhängigen Funktionen. Z. Angew. Math. Mech. **35**, 161–169 (1955). (English, French and Russian summaries)

Let $\{\varphi_n(x)\}$ $(n=0, 1, \cdots)$ be a sequence of real, linearly independent functions of the k variables $x=(x_1, \cdots, x_k)$, defined over a region B. Given a function $f(x)$, under some general assumptions there exists for each n among all linear combinations $P(x)=\sum_{i=0}^{n-1} c_i\varphi_i(x)$ a unique one that minimizes $\int_B [f(x)-P(x)]^2dx$. If $c_i=A_i^{(n)}$ for the minimizing $P(x)$ then $A_i^{(n)}=\int_B f(x)\Phi_{in}(x)dx$, where $\Phi_{in}(x)=\sum_{j=0}^{n-1}\alpha_{ij}^{(n)}\varphi_j(x)$ (for suitably chosen coefficients $\alpha_{ij}^{(n)}$, independent of $f(x)$). Orthogonality properties are found for the functions Φ_{ij}, and relations between $\{\alpha_{ij}^{(n)}\}$ and $\{A_i^{(n)}\}$ determined. The general theory is then applied to the one-dimensional case $\varphi_i(x)=x^i$ with $B=[0, 1]$ and $[-1, 1]$ respectively. For the first case it is shown that if $f(x)=\sum_0^\infty c_nx^n$ has a convergence radius $r>1$ then (see above notation) $\lim_{n\to\infty} A_i^{(n)}=c_i$. *I. M. Sheffer.*

Kaplan, Wilfred. Approximation by entire functions. Michigan Math. J. **3**, 43–52 (1955).

The author gives first a concise version (suggested by M. Brelot) of the proof of Carleman's theorem: If $Q(x)$ is continuous on $-\infty<x<\infty$ and $E(x)$ is continuous and >0, then there is an entire function $f(z)$ satisfying

$$(1) \qquad |f(x)-Q(x)|<E(x).$$

He shows that, for $Q\in C^1(-\infty, \infty)$, $f(z)$ can be chosen so that simultaneously (2) $|f'(x)-Q'(x)|<E(x)$. This theorem is applied to show that a homeomorph of the real axis $z=Q(t)$ $(Q\in C^1(-\infty, \infty);\ Q'(t)\neq 0)$ can be approximated by a homeomorph of the real axis $z=f(t)$, where f is an entire function satisfying (1) and (2).

An analogue of Carleman's theorem is proved where the real axis is replaced by a set of open Jordan arcs in a region D such that any compact set inside D meets only a finite number of the arcs and with entire functions replaced by functions regular in D. The final section shows how these approximation theorems can be applied to find solutions of the Dirichlet problem. Typical is the theorem: Let $Q(z)$ be a real-valued measurable function defined on $|z|=1$. There is a function $u(z)$ harmonic in $|z|<1$ such that $u(re^{i\theta})\to Q(e^{i\theta})$ as $r\to 1$ almost everywhere in $0\leq\theta\leq 2\pi$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Bernšteĭn, S. N. An application of a limit law of the theory of best approximation. Dokl. Akad. Nauk SSSR (N.S.) **102**, 435–436 (1955). (Russian)

Theorem. Let $H(z)$ be an even entire function of order zero, positive and monotonely increasing for non-negative x. If there is an infinity of n such that $|f^{(n)}(x)|<p^nH(x)$ $(-\infty<x<\infty)$, then $f(x)$ is an entire function of order one, type $\leq p$.

The proof is based on a theorem of the author [Collected works, v. II, Izdat. Akad. Nauk SSSR, Moscow, 1954, p. 392; MR **16**, 433] expressing the best approximation to $f(x)$ by entire functions of order one type q as the limit of the best approximation by polynomials of degree n in the interval $|x|\leq n/q$. *W. H. J. Fuchs.*

Džrbašyan, M. M. Some questions of the theory of weighted polynomial approximations in a complex domain. Mat. Sb. N.S. **36(78)**, 353–440 (1955). (Russian)

This article falls into two parts. The first part reports, with proofs, the results of the author on approximation by polynomials on curves (and other sets) extending to infinity. Both uniform approximation with a wieght-function and mean square approximation with a weight-function are considered. The theorems of the first part have been announced previously [Dokl. Akad. Nauk SSSR (N.S.) **62**, 581–584 (1948); **66**, 1037–1040; **67**, 15–18 (1949); Uspehi Mat. Nauk (N.S.) **5**, no. 3(37), 194–198 (1950); MR **10**, 364; **11**, 94, 95; **12**, 16]. In Part II $|P_n(z_0)|$ is estimated for polynomials of the nth degree satisfying an inequality $w(\zeta)|P_n(\zeta)|\leq 1$ for all ζ in a set E of the z-plane. Here z_0 must lie in a region containing E and depending on n. The weight-function w is subject to various regularity restrictions. The set E is taken in turn as the real line, two half-lines meeting at an angle α, the interior of an angle.

Cauchy's formula then leads to estimates of the derivatives of $P_n(\zeta)$ on E. The exact statements are too long for quotation. An interesting feature is that the estimates are much better than those available for polynomials bounded in a bounded set. From these estimates direct and inverse theorems are obtained connecting the best approximation $E_n(f)$ of a function $f(z)$,

$$E_n(f)=\inf_{P_n}\sup_{\zeta\in E} w(\zeta)|f-P_n(\zeta)|,$$

with the differential properties of f. *W. H. J. Fuchs.*

Salem, R. On a problem of Littlewood. Amer. J. Math. **77**, 535–540 (1955).

Raffinement des méthodes déjà utilisées par l'auteur [Duke Math. J. **8**, 269–272 (1941); MR **3**, 108] pour démontrer le théorème de Menchoff sur la convergence presque-partout des séries orthogonales. Entre autres applications, une solution partielle est donnée à un probleme de Littlewood: si $\{m_n\}$ est une suite croissante d'entiers, avec $\log m_n=O(\log n)$, on a

$$\limsup (\log n)^{-\frac{1}{2}}\int_0^{2\pi}|\cos m_1x+\cdots+\cos m_nx|dx>0.$$

J. P. Kahane (Montpellier).

Fine, N. J. Cesàro summability of Walsh-Fourier series. Proc. Nat. Acad. Sci. U.S.A. **41**, 588–591 (1955).

The author demonstrates the almost everywhere (C, α), $\alpha>0$, summability of Walsh-Fourier series. The proof depends on the following lemma. Let C be a measurable subset of $[0, 1]$ and $D(x)$ the distance from x to C. If the sequence $h_1\geq h_2\geq\cdots>0$ satisfies $h_{j+r}\geq qh_j$ for some $r\geq 1$, $0<q<1$, for all $j\geq 1$, then $\sum D(x\pm h_j)/h_j<\infty$, for each fixed but arbitrary choice of signs, for almost all $x\in C$. *P. Civin* (Eugene, Ore.).

Hale, Jack K. Evaluations concerning products of exponential and periodic functions. Riv. Mat. Univ. Parma **5**, 63–81 (1954).

The author considers the family C_ω of functions which are finite sums of functions of the form $e^{\alpha x}\varphi(x)$, x real, α any complex number, φ any complex periodic function of period $T=2\pi/\omega$. For any $f\in C_\omega$ there is a unique (finite) series $f=\sum e^{\alpha_jx}\varphi_j(x)$, the α_j being incongruent with each other mod ωi; if the Fourier series of the φ_j are then formally replaced in the above expression and the constant terms are collected together, their sum is called "mean value" of f. The following results are proved: (i) $f\in C_\omega$ has a primitive $F\in C_\omega$ if and only if its mean value is 0; (ii) if furthermore all the φ_j have mean value 0, $F=\sum\int_{\xi_i}^x f_i(x)dx$ with $f_i\in C_\omega$, $\xi_i\in[0, T]$; (iii) if the assumption of (i) is satisfied, there are functions $N(\alpha_j, T)$ such that $|F|\leq\sum N(\alpha_j, T)\int_0^T|\varphi_j(x)|dx$. *J. L. Massera.*

Hayes, Wallace D. Neo-Schlömilch series. J. Math. Phys. **34**, 129–132 (1955).

The following theorem is proved. Let $f(x)$ be an arbitrary function with a derivative $f'(x)$ that is continuous and of bounded variation in the closed interval $0\leq x\leq\pi$. Then

$$f(x)=\tfrac{1}{2}a_0J_0(kx)+\sum_{m=1}^{\infty}a_mJ_0[(k^2+m^2)^{\frac{1}{2}}x] \quad (0\leq x\leq\pi,\ k\geq 0),$$

in which the coefficients a_m are given by

$$a_m=\frac{2}{\pi}\int_0^{\pi}f(x)C_m(x,k)dx,$$

where the function C_m is defined by

$$C_m(x,k)=\frac{(-1)^m x\cosh[k(\pi^2-x^2)^{\frac{1}{2}}]}{(\pi^2-x^2)^{\frac{1}{2}}}+\int_x^{\pi}\frac{mx\cosh[k(t^2-x^2)^{\frac{1}{2}}]\sin mt}{(t^2-x^2)^{\frac{1}{2}}}dt.$$

C. J. Bouwkamp (Eindhoven).

Atkinson, F. V. On lacunary and other orthogonal polynomials. Univ. Nac. Tucumán. Rev. Ser. A. **10**, 95–110 (1954).

Systems of polynomials are considered arising by orthogonalization of a given sequence of powers of x with respect to a distribution $d\sigma(x)$ $(a\leq x\leq b)$. The case $1, x^2, x^3, x^4, \cdots$ is investigated in some detail; the resulting polynomials have many characteristics similar to those in the classical case. Special distributions are discussed, for instance $d\sigma(x)=dx$ or $e^{-x}dx$. Indications of other "lacunary" polynomials and their properties are given. *G. Szegő* (Stanford, Calif.).

Trigonometric Series and Integrals

Chow, Hung Ching. Some new criteria for the absolute summability of a Fourier series and its conjugate series. J. London Math. Soc. **30**, 439–448 (1955).

Let $W_p(t)=\left[(2\pi)^{-1}\int_{-\pi}^{\pi}|f(x+t)-f(x)|^p dx\right]^{1/p}$ and

$$W(x;t)=|f(x+t)-f(x)|.$$

The following conditions are shown to be sufficient for the almost everywhere summability $|C,\alpha|$ of the Fourier series or the conjugate Fourier series of $f(x)$: (1) If $1\leq p\leq 2$, then $W_p(t)=O\{(\log|t|^{-1})^{-1-\delta}\}$, $\delta>0$, as $t\to 0$, or $\int_{-\pi}^{\pi}W_p(t)|t|^{-1}dt<\infty$ is sufficient for $\alpha>1/p$. (2) If $1\leq p\leq 2$, then $W_p(t)=O\{(\log|t|^{-1})^{-1-1/p-\delta}\}$, as $t\to 0$ $\delta>0$ is sufficient for $\alpha=1/p$. (3) $W(x;t)=O\{(\log|t|^{-1})^{-1-\delta}\}$, $\delta>0$, uniformly in x, as $t\to 0$, is sufficient for $\alpha=\frac{1}{2}$.

P. Civin (Eugene, Ore.).

Izumi, Shin-ichi, and Kinukawa, Masakiti. On the strong summability of the derived Fourier series. II. Proc. Japan Acad. **31**, 107–110 (1955).

[For part I see Kinukawa, same Proc. **30**, 801–804 (1954); MR **16**, 1100.] Let $f(t)$ be a continuous function of bounded variation with period 2π and let

$$g(u)=f(x+u)-f(x-u)-2us.$$

If $\int_0^t|dg(u)|=O[t/(\log t^{-1})^{\alpha}]$ as $t\to 0$ with $\alpha>\frac{1}{2}$, then the derived Fourier series for $f(t)$ is H_2-summable at $t=x$ to s, i.e. if $t_n(x)$ is the nth partial sum of the derived Fourier series $\lim_{n\to\infty}n^{-1}\sum_{\nu=1}^{n}|t_\nu(x)-s|^2=0$.

P. Civin (Eugene, Ore.).

Rudin, Walter. Nonanalytic functions of absolutely convergent Fourier series. Proc. Nat. Acad. Sci. U.S.A. **41**, 238–240 (1955).

Soit A (resp. A^+) l'ensemble des fonctions

$$f(\theta)=\sum_{-\infty}^{\infty}a_n\exp(in\theta)\ (\text{resp. }\sum_0^{\infty})$$

avec $\sum|a_n|<\infty$. L'auteur indique des conditions suffisantes portant sur φ pour qu'existe $f\in A^+$ telle que $\varphi\circ f\notin A$: par exemple, si $\varphi(w)$ est continue dans le demi plan $u\geq 0$ et analytique pour $u>0$, il suffit $\varphi'(u)$ ne soit pas bornée pour $u>0$. Application: $\varphi(w)=\sqrt{w}$; il existe $f\in A$, positive, telle que $\sqrt{f}\notin A$ [ce qui résulte aussi, d'un résultat du référent, C. R. Acad. Sci. Paris **240**, 36–37 (1955); MR **16**, 467].

J. P. Kahane (Montpellier).

Džvaršeĭšvili, A. G. On an inequality of A. Zygmund for functions of two variables. Soobšč. Akad. Nauk Gruzin. SSR **15**, 561–568 (1954). (Russian)

It is well known [see A. Zygmund, Prace Mat.-Fiz. 33, 125–132 (1924)] that if a periodic $f(x)$ of period 2π has modulus of continuity $\omega(\delta)$, then the conjugate function

$$g(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\tfrac{1}{2}f(x+t)\cot\tfrac{1}{2}t\,dt$$

has a modulus of continuity majorized by

$$K\Big[\int_0^{\delta}t^{-1}\omega(t)dt+\delta\int_{\delta}^{\pi}t^{-2}\omega(t)dt\Big],$$

where K is an absolute constant. An analogous result is obtained for conjugate functions

$$g(x,y)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\tfrac{1}{2}f(x+s,y+t)\cot\tfrac{1}{2}t\cot\tfrac{1}{2}s\,ds dt$$

of two variables. *A. Zygmund* (Chicago, Ill.).

Levitan, B. M. On summability of multiple series and Fourier integrals. Dokl. Akad. Nauk SSSR (N.S.) **102**, 1073–1076 (1955). (Russian)

Suppose that $f(x)=f(x_1,x_2,\cdots,x_n)\in L(E_n)$ $(n>1)$. Write

$$E(\alpha)=E(\alpha_1,\cdots,\alpha_n)=(2\pi)^{-n}\int_{E^n}f(x)e^{-i(x,\alpha)}dx,$$

where $(\alpha,x)=\alpha_1x_1+\cdots+\alpha_nx_n$, $dx=dx_1\cdots dx_n$, and

$$R_s(x,\mu)=\frac{1}{\Gamma(s+1)}\int_{|\alpha|\leq\mu}\Big(1-\frac{|\alpha|^2}{\mu^2}\Big)^s E(\alpha)e^{i(x,\alpha)}d\alpha,$$

where $s\geq 0$, $|\alpha|^2=\alpha_1^2+\cdots+\alpha_n^2$, $d\alpha=d\alpha_1\cdots d\alpha_n$. The proof of the following result is given in a special case. 1) If $n=2k+1$, $k\geq 1$, $f(x)$ is in $L_2(E_n)$, is continuous and has everywhere continuous derivatives up to order $\lambda>0$ inclusive, quadratically integrable over E_n, then

$$\lim_{\mu\to\infty}R_{k-1}(x,\mu)=\frac{1}{\Gamma(k-\lambda+1)}f(x),$$

uniformly in every bounded domain; in particular, for $\lambda=k$ we have convergence (a result is stated also for the case of $n=2k$ even, but it is conjectured that it is not best possible). The following two theorems are stated without proof. 2) Suppose that $f(x)\in L_2(E_n)$ $(n=2k+1, k\geq 1)$, and that f has in the neighborhood of x_0 continuous derivatives up to order $r>0$ inclusive. Then at x_0 we have

$$(\dagger)\qquad \lim_{\mu\to\infty}\frac{\partial^r R_s(x,\mu)}{\partial x_1^{\lambda_1}\cdots\partial x_n^{\lambda_n}}=\frac{1}{\Gamma(s+1)}\frac{\partial^r f(x)}{\partial x_1^{\lambda_1}\cdots\partial x_n^{\lambda_n}}$$

with $s=k+\tau$. 3) Suppose that $f(x) \in L_2(E_n)$ $(n=2k+1, k\geq 1)$, and that f has everywhere in E_n continuous derivatives with integrable square up to order r_0 inclusive. Suppose also that (in the neighborhood of x_0) f has continuous derivatives up to order $r=r_0+r_1$ inclusive. Then at x_0 we have (†), where this time $s=k+r_1$.

A. Zygmund (Chicago, Ill.).

Ilieff, Ljubomir. Über trigonometrische Integrale, welche ganze Funktionen mit nur reellen Nullstellen darstellen. Acta Math. Acad. Sci. Hungar. **6**, 191–194 (1955). (Russian summary)

German version of a paper already reviewed [Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 147–153 (1954); MR **16**, 1018]. *R. P. Boas, Jr.* (Evanston, Ill.).

Wiener, Norbert. Optics and the theory of stochastic processes. J. Opt. Soc. Amer. **43**, 225–228 (1953).

Let $g(w)$ be a matrix function of a real parameter w, Hermitian for each value of w. Theorem: The condition

$$\int_{-\infty}^{\infty} |\log \det (g(w))|(1+w^2)^{-1}dw<0 \tag{1}$$

is necessary and sufficient for $g(w)$ to be expressible in the form

$$g(w)=A(w)\cdot A^{+}(w), \tag{2}$$

where $A(w)$ is the matrix

$$A(w)=\int_0^{\infty} e^{-iwt}dK(\tau) \tag{3}$$

whose spectrum extends only over positive values of τ. No proof is given.

Applications of this theorem to optics and to quantum mechanics are briefly discussed. The idea is that $A(w)$ is the most general form of a frequency-spectrum for a wave-filter or scatterer whose behavior is causal, not allowing any response to occur before a signal is received. Then $g(w)$ given by (2) is the power-spectrum transmitted by such a device when the incident power-spectrum is uniform. Condition (1) is a limitation on the possible power-spectra obtainable in this way. *F. J. Dyson.*

See also: Hale, pp. 31.

Integral Transforms, Operational Calculus

★**Starkey, B. J. Laplace transforms for electrical engineers.** Iliffe & Sons, Ltd., London, 1954; Philosophical Library, New York, 1955. 279 pp. $10.00.

As the title implies, this book is addressed to engineers rather than to mathematicians. The mathematical content of the book is of a descriptive nature. Vector algebra is frequently employed, with the intention of clarifying concepts for the reader whose mathematical background is limited to calculus and the elements of vector analysis. The three principal topics consist of basic properties of the Laplace transformation, the calculus of functions of a complex variable, and applications to problems in electric circuits. The number of pages written on each of these topics is about the same, but the treatment of each topic is distributed through the book. The descriptive nature of the exposition extends to incomplete statements of mathematical results and definitions. Except for one example on the telegraph equation, the applications deal with ordinary differential equations with constant coefficients and systems of such equations. The appendices contain notes on non-elementary functions, and a table of transforms. *R. V. Churchill.*

Vasilache, Sergiu. Sur quelques formules de la théorie de Laplace à deux variables. Com. Acad. R. P. Române **2**, 193–197 (1952). (Romanian. Russian and French summaries)

Expression of the double Laplace transform of $x^{\alpha}y^{\beta}f(x, y)$ in terms of the double Laplace transform of $f(x, y)$.

A. Erdélyi (Pasadena, Calif.).

Chak, A. M. Some theorems in operational calculus. II. Ann. Univ. Lyon. Sect. A. (3) **17**, 11–17 (1954).

Continuation of an earlier paper [same Ann. **16**, 53–62 (1954); MR **15**, 869]. Formal extensions from Hankel transforms to general transforms and from one variable to several variables. *A. Erdélyi* (Pasadena, Calif.).

Special Functions

Castoldi, Luigi. Di una esatta formula generalizzata di Stirling. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 152–156 (1955).

The author's "exact formula" is the well-known Stirling series (the leading term of which is Stirling's formula).

A. Erdélyi (Pasadena, Calif.).

Hallén, Erik. Further investigations into iterated sine- and cosine-integrals and their amplitude functions with reference to antenna theory. Kungl. Tekn. Högsk. Handl. Stockholm no. **89**, 44 pp. (1955).

Let

$$L(x)=\int_0^x (1-e^{-jt})\xi^{-1}d\xi, \quad L^{01}(x)=\int_0^x L(\xi)\xi^{-1}d\xi,$$

$$L^{11}(x)=\int_0^x e^{jt}L(\xi)\xi^{-1}d\xi, \quad L^{12}(x)=\int_0^x e^{jt}L(2\xi)\xi^{-1}d\xi,$$

$$L^{001}(x)=\int_0^x L^{01}(\xi)\xi^{-1}d\xi, \quad L^{011}(x)=\int_0^x L^{11}(\xi)\xi^{-1}d\xi,$$

$$L^{101}(x)=\int_0^x e^{jt}L^{01}(\xi)\xi^{-1}d\xi, \quad L^{111}(x)=\int_0^x e^{-jt}L^{11}(\xi)\xi^{-1}d\xi;$$

$$l(x)=\int_x^{\infty} e^{-j(t-x)}\xi^{-1}d\xi, \quad l^{01}(x)=\int_x^{\infty} l(\xi)e^{-j(t-x)}\xi^{-1}d\xi,$$

$$l^{11}(x)=\int_x^{\infty} l(\xi)\xi^{-1}d\xi, \quad l^{12}(x)=\int_x^{\infty} l(2\xi)e^{-j(t-x)}\xi^{-1}d\xi,$$

$$l^{001}(x)=\int_x^{\infty} l^{01}(\xi)e^{-j(t-x)}\xi^{-1}d\xi, \quad l^{011}(x)=\int_x^{\infty} l^{11}(\xi)\xi^{-1}d\xi,$$

$$l^{101}(x)=\int_x^{\infty} l^{01}(\xi)\xi^{-1}d\xi \text{ and } l^{111}(x)=\int_x^{\infty} l^{11}(\xi)e^{-j(t-x)}\xi^{-1}d\xi.$$

When x is real, any of these functions have real and imaginary parts which are denoted by $L(x)=C(x)+jS(x)$ and $l(x)=c(x)+js(x)$ with corresponding subscripts. These functions play a dominant role in the author's antenna theory, and numerical tables of some of them are available. In this paper the author derives power-series and asymptotic expansions for all of these functions, shows their interrelation, and obtains alternative expressions in the form of definite integrals. The following tables are included: Table I: $x=0(0.01)0.2(0.2)14.0$; 5 or 6D; $C^{11}(x)$, $S^{11}(x)$, $C^{12}(x)$, $S^{12}(x)$. Table II: $x=0(0.01)0.20$; 6D; $C^{01}(x)$, $S^{01}(x)$. Table III: $x=0(0.01)0.2(0.2)14.0$; 5D; C^{001}, S^{001}, C^{101}, S^{101}, C^{011}, S^{011}, C^{111}, S^{111}. Table IV:

$x=0(0.01)0.2(0.1)14.0(0.2)28(1)30$; 6D; $c(x)$, $s(x)$; and the functions $c(x)+\log x+\gamma$, $s(x)+\frac{1}{4}\pi$, where γ is Euler's constant, but only up to $x=14$. Table V: $x=0(0.01)0.20(0.2)14.0(1)30$; 4,5 or 6D; c^{01}, s^{01}, c^{11}, s^{11}, c^{12}, s^{12}. Table VI: same x values as in table V; 4 or 5D; c^{001}, s^{001}, c^{101}, s^{101}, c^{011}, s^{011}, c^{111}, s^{111}.

C. J. Bouwkamp (Eindhoven).

Denisyuk, I. M. Some relations which contain the normalized Laguerre polynomials and analogues to them. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 324–326 (1954). (Ukrainian. Russian summary)

The polynomials M_n were introduced in earlier papers [same Dopovidi **1954**, 79–81, 165–167, 239–242; MR **16**, 694]. Here the author writes down some more recurrence and differentiation formulas, and evaluates some infinite integrals containing L_n, M_n, and derivatives.

A. Erdélyi (Pasadena, Calif.).

Denisyuk, I. M. New polynomials analogous to the Laguerre polynomials. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 327–330 (1954). (Ukrainian. Russian summary)

New polynomials $\mathfrak{M}_n(x)$ may be defined by the generating function

$$\frac{1-t}{(1+t)^2}\exp\left(-\frac{xt}{1-t}\right)=\sum_{n=0}^{\infty}t^n\mathfrak{M}_n(x).$$

Their connection with the polynomials $\mathfrak{M}_n$ [same Dopovidi **1954**, 79–81, 165–167, 239–242; MR **16**, 694; see also the paper reviewed above] is shown and their coefficients are tabulated. *A. Erdélyi* (Pasadena, Calif.).

Palamà, Giuseppe. Integrali generali delle equazioni differenziali cui soddisfano polinomi che generalizzano altri classici ed i loro associati. Boll. Un. Mat. Ital. (3) **10**, 201–207 (1955).

In an earlier paper [Riv. Mat. Univ. Parma **4**, 363–386 (1953); MR **16**, 470] the author defined polynomials $R_{n,p}(x)$ generalizing a given system of polynomial solutions of a recurrence relation. In the present note he studies in particular the polynomials generalizingLaguerre and Hermite polynomials, obtains formulas for the derivatives of, and differential equations satisfied by, these polynomials. The general solutions of the differential equations are also given.

A. Erdélyi (Pasadena, Calif.).

Palamà, Giuseppe. Sviluppo di alcuni polinomi che generalizzano altri classici ed i loro associati e relazioni tra essi. Boll. Un. Mat. Ital. (3) **10**, 233–238 (1955).

The author gives expansions of the polynomials investigated in the note reviewed above as well as of polynomials generalizing ultraspherical polynomials; and he establishes various relations involving these polynomials. *A. Erdélyi* (Pasadena, Calif.).

Brafman, Fred. Series of products of Gegenbauer polynomials. Math. Z. **62**, 438–442 (1955).

The author finds a generating function for products of three Gegenbauer polynomials (or ultraspherical polynomials) which are of the same index but which are functions of different variables. This extends a result of Vinti [Proc. Amer. Math. Soc. **2**, 19–23 (1951); MR **13**, 233], the new generating function being a power of Vinti's. *N. D. Kazarinoff* (Lafayette, Ind.).

Gatteschi, Luigi. Sulla rappresentazione asintotica delle funzioni di Bessel di uguale ordine ed argomento. Ann. Mat. Pura Appl. (4) **38**, 267–280 (1955).

The author applies van der Corput's theory of enveloping series to Sommerfeld's integral representation of $H_\nu^{(1)}(\nu)$ to obtain asymptotic expansions of $J_\nu(\nu)$, $Y_\nu(\nu)$ in descending powers of $\nu^{\frac{1}{3}}$, together with precise estimates of the remainder. He also establishes

$$J_\nu(\nu)=2^{-\frac{1}{3}}3^{-\frac{1}{6}}\pi^{-1}\nu^{-\frac{1}{3}}\Gamma\left(\frac{1}{3},\left(\frac{2\pi}{\sqrt{3}}\right)^3\frac{\nu}{6}\right)+\varrho,$$

where $\Gamma(\alpha, x)$ is the complementary incomplete gamma function and

$$|\varrho|<\frac{1}{\nu\pi}\exp\left(-\frac{\nu\pi}{\sqrt{3}}\right)+\frac{1.4}{\pi}\left(\frac{6}{\nu}\right)^{\frac{5}{3}}.$$

A. Erdélyi (Pasadena, Calif.).

Muller, G. M. On certain infinite integrals involving Bessel functions. J. Math. Phys. **34**, 179–181 (1955).

The author points out that it follows from the known evaluation of

$$\int_0^\infty K_p(t)I_q(tz)t^r\,dt$$

as a hypergeometric series that this integral may be expressed as a linear combination, with coefficients rational in z, of elliptic integrals of the first and second kinds provided that q and $\frac{1}{2}(q+r\pm p)$ are non-negative integers. He gives two examples: (i) $p=q=r=0$, (ii) $p=q=0$, $r=2$. *A. Erdélyi* (Pasadena, Calif.).

Müller, R. Spezielle Integrale mit Zylinder- und Kugelfunktionen. Z. Angew. Math. Mech. **35**, 62–64 (1955).

Let $f(z)$ be a solution of $f''+A(z)f'+B(z)f=0$, $v(z)$ any function, and $W(f, v)$ the Wronskian $fv'-f'v$, then the following identity obtains:

$$\int_{z_0}^{z}\exp\left\{\int_{z_0}^{s}A(u)du\right\}f(z)\left\{v''(z)+A(z)v'(z)+B(z)v(z)\right\}dz=\left[\exp\left\{\int_{z_0}^{s}A(u)du\right\}W(f,v)\right]_{s=z_0}^{s=z}.$$

In applying this identity to $f(z)=Z_\nu(z)$, $z^nZ_\nu(z)$, and $K_\nu(z)$, where Z_ν and K_ν denote Bessel and Legendre functions, respectively, the author obtains, by taking for v simple functions of z, various kinds of explicit expressions for indefinite integrals containing Bessel or Legendre function in the respective integrands. *C. J. Bouwkamp.*

Schubert, Andreas. Beiträge zur Integration von Funktionen, in denen Produkte von Zylinderfunktionen auftreten. Wiss. Z. Tech. Hochsch. Dresden **2**, 437–440 (1953).

In dieser Arbeit werden Rekursionsformeln für einige Integrale abgeleitet, in deren Integranden Produkte von Zylinderfunktionen auftreten. Wie üblich werden Z_p und $\bar{Z}_p$ als Abkürzungen für

$$Z_p(\alpha x)=c_1J_p(\alpha x)+c_2N_p(\alpha x),$$
$$\bar{Z}_p(\alpha x)=\bar{c}_1J_p(\alpha x)+\bar{c}_2N_p(\alpha x)$$

verwendet, wobei die (reellen oder komplexen) Konstanten c_1, c_2, $\bar{c}_1$, $\bar{c}_2$ als vom Index p unabhängig angenommen werden. Es werden Rekursionsformeln für die Integrale

$$A_{n,p,q}=\int x^{2n+1}Z_{p-q}\bar{Z}_{p+q}\,dx,$$
$$B_{n,p,q}=\int x^{2n+2}Z_{p-q}\bar{Z}_{p+q+1}\,dx$$

hinsichtlich q sowie für die Integrale $A_{n,p,0}$ und $B_{n,p,0}$ hinsichtlich n angegeben. Es ergeben sich Summen von

Ausdrücken nach Art der Integranden. Verf. weist noch auf die Arbeit von J. Picht hin [Z. Angew. Math. Mech. 29, 155–157 (1949); MR 12, 25].

R. Gran Olsson (Zbl 51, 53).

Conolly, B. W. Two integrals involving modified Bessel functions of the second kind. Proc. Glasgow Math. Assoc. 2, 147–148 (1955).

Generalization of a formula by Nicholson, and its Fourier inversion [see also Erdélyi et al., Tables of integral transforms, vol. 2, McGraw-Hill, New York, 1954, 19.6 (45) and 19.7 (23); MR 16, 468]. *A. Erdélyi.*

MacRobert, T. M. Integrals involving products of modified Bessel functions of the second kind. Proc. Glasgow Math. Assoc. 2, 129–131 (1955).

The author evaluates

$$\int_0^\infty e^{-st}t^{k-1}K_m(t)K_n(z/t)dt$$

as a sum of infinite series of powers of z, the coefficients being hypergeometric functions of s. *A. Erdélyi.*

MacRobert, T. M. Integrals involving E-functions and associated Legendre functions. Proc. Glasgow Math. Assoc. 2, 127–128 (1955).

Evaluation of the integrals

$$\int_1^\infty E\left(p;\alpha_r:q;\varrho_s:\frac{z}{t-1}\right)(t-1)^{l-1}(t^2-1)^{\pm m/2}P_n^{-m}(t)dt$$

in terms of E-functions, together with a special case involving a product of Legendre functions. *A. Erdélyi.*

Ragab, F. M. A product of two E-functions expressed as a sum of two E-functions. Proc. Glasgow Math. Assoc. 2, 124–126 (1955).

The first formula proved in this note is in G. N. Watson's "Bessel functions" [Cambridge, 1922, p. 147, equation (5)], the second is an immediate consequence [Erdélyi et al., Higher transcendental functions, v. 1, McGraw-Hill, New York, 1953, 5.6 (66); MR 15, 419]; and the integral deduced from these has been evaluated by C. S. Meijer [Nederl. Akad. Wetensch. Proc. 44, 81–92 (1941); MR 2, 287]. *A. Erdélyi* (Pasadena, Calif.).

Rathie, C. B. Some results involving hypergeometric and E-functions. Proc. Glasgow Math. Assoc. 2, 132–138 (1955).

Let $\phi(p)\doteq h(x)$ and $g(p)\fallingdotseq h(x)$ stand, respectively for

$$\phi(p)=p\int_0^\infty e^{-px/2}(px)^{m-1/2}W_{k,m}(px)h(x)dx$$

and

$$g(p)=p\int_0^\infty e^{-px}h(x)dx.$$

The author expresses $\phi(p)$ as an integral involving $f(x)$ when $\phi(p)\doteq h(x)$ and either $f(p)\fallingdotseq x^{\frac{1}{2}}h(x)$ or $p^{2-k}h(p)\fallingdotseq g(x)$ and $p^{2-n}g(p)\fallingdotseq f(x)$. Both results are used to evaluate some integrals. *A. Erdélyi* (Pasadena, Calif.).

Chaundy, T. W. A hypergeometric identity. Quart. J. Math. Oxford Ser. (2) 6, 132–134 (1955).

A hypergeometric identity proved in an earlier paper [J. London Math. Soc. 26, 42–44 (1951); MR 12, 410] is first simplified in a special case ($ab=eh$) and then extended to the first of Appell's four hypergeometric series of two variables. *A. Erdélyi* (Pasadena, Calif.).

Toscano, Letterio. Su una formula limite tra funzioni ipergeometriche di Kummer e funzioni del cilindro parabolico. Boll. Un. Mat. Ital. (3) 10, 239–243 (1955).

Proof and applications of the formula

$$\lim_{\alpha\to\infty}\frac{\Gamma(\alpha+h+\nu+1)}{\alpha^{\nu/2}\Gamma(\alpha+h+1)}{}_1F_1\left(-\nu;\alpha+h+1;\alpha\frac{\alpha^k-x}{\alpha^k-t}\right)$$
$$=\exp\left[\tfrac{1}{4}(x-t)^2\right]D_\nu(x-t).$$

A. Erdélyi (Pasadena, Calif.).

Kobayashi, Shigehiro. Some coefficients of the series expansion of the TFD function. J. Phys. Soc. Japan 10, 824–825 (1955).

The function (Thomas-Fermi-Dirac) under consideration is a solution of $\psi''=x(\varepsilon+(\psi/x)^{\frac{1}{2}})^3$, $\psi(0)=1$.

Ordinary Differential Equations

Aquaro, Giovanni. Sul teorema di esistenza di Caratheodory per i sistemi di equazioni differenziali ordinarie. Boll. Un. Mat. Ital. (3) 10, 208–212 (1955).

In proving the existence of a solution of the equation $y'=f(x,y)$ for $a\leq x\leq b$, it is often assumed (A) that f is dominated by a summable function. Here the author lets $y(x)$ range over the space C of functions continuous for $a\leq x\leq b$, and assumes that $f(t,y(t))$ is summable. Then an assumption weaker than (A) is that the set of functions $\int_a^x f(t,y(t))dt$ is equiabsolutely continuous. A still weaker adequate assumption is that there exists for $0\leq x<\infty$ a positive continuous non-summable function $\varphi(x)$ such that the set of functions $\int_a^x \varphi(\|y(t)\|)\|f(t,y(t))\|dt$ is equiabsolutely continuous. The Euclidean norm is used for vectors. *F. A. Ficken* (Knoxville, Tenn.).

Nalšul', A. B. Linear functional problems. Dokl. Akad. Nauk SSSR (N.S.) 102, 21–23 (1955). (Russian)

The author considers the vector differential equation

$$\frac{d\mathbf{r}(x)}{dx}=A(x)\mathbf{r}(x)+\mathbf{b}(x), \tag{1}$$

where $\mathbf{r}(x)$ and $\mathbf{b}(x)$ are vectors in an n-dimensional space and $A(x)$ is an $n\times n$ matrix; he seeks solutions satisfying the restrictive condition

$$Q\mathbf{r}(a)+\int_a^b dG(x)\mathbf{r}(x)=\varphi, \tag{2}$$

where Q and $G(x)$ are $n\times n$ matrices, $G(x)$ is of bounded variation in (a,b) and $G(b)=O$.

He shows that if $R(x)$ is the fundamental matrix solution of the associated homogeneous equation of (1), satisfying $R(a)=E$, where E is the unit matrix, and $\varrho(x)$ is a solution of (2) satisfying $\varrho(a)=\mathbf{O}$, then the number of linearly independent solutions of the system (1), (2) is the same as for

$$U\mathbf{r}(a)=\psi, \tag{3}$$

where $U=Q+\int_a^b dG(x)R(x)$, $\psi=\varphi-\int_a^b dG(x)\varrho(x)$.

Various other conditions equivalent to (2) are derived, and are used to obtain an initial condition equivalent to (2). Conditions for the existence and uniqueness of a solution of the system bearing directly on the matrices $A(x)$, Q and $G(x)$ are obtained; these are too complicated to reproduce here. An application is given to a special system of ordinary differential equations.

F. Smithies (Cambridge, England).

Bellman, Richard. On perturbation methods involving expansions in terms of a parameter. Quart. Appl. Math. **13**, 195–200 (1955).

From the author's summary: "It is shown by means of some examples from the theories of linear algebraic equations, linear integral equations and nonlinear differential equations that the effectiveness of the method of expanding a solution in a power series in terms of a parameter may in many cases be greatly increased by expanding in terms of a suitably chosen function of the parameter. This is particularly the case when the physical setting of the problem allows only positive values of the parameter to enter." *N. Levinson* (Cambridge, Mass.).

Feščenko, S. F. Estimate of the error in the asymptotic solution of an infinite system of differential equations with slowly varying coefficients. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 211–216 (1955). (Ukrainian. Russian summary)

The same infinite system of differential equations is considered as that given in an earlier paper by the author [same Dopovidi **1954**, 82–86; MR **16**, 1111]. The present paper establishes the convergence of the formal solution of the given system of differential equations and gives an estimate of the error of the mth approximation for the solution. *H. P. Thielman* (Ames, Iowa).

García, Godofredo. On the integration and properties of a differential equation. Actas Acad. Ci. Lima **17**, 39–50 (1954). (Spanish)

The equation $y''+G^2(x)y=0$ is considered. The reviewer has not been able to understand this paper, perhaps because of several misprints. In any case some of the results, as printed (for instance, the last example), are false. *J. L. Massera* (Montevideo).

Ura, Taro, et Hirasawa, Yoshikazu. Sur les points singuliers des équations différentielles admettant un invariant intégral. Proc. Japan Acad. **30**, 726–730 (1954).

It is shown that a system $x=X(x, y)$, $y=Y(x, y)$ having a positive invariant integral can have as isolated singularities (points at which $X=Y=0$) only multiple saddle points or centers. *W. Kaplan* (Ann Arbor, Mich.).

Hale, Jack K. On boundedness of the solutions of linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma **5**, 137–167 (1954).

Consider a matrix differential equation $Y'=AY+\lambda\Phi Y$, A, Φ, Y being n^2-matrices, λ complex number, Φ L-integrable and periodic in x (independent variable) with period $T=2\pi/\omega$, $\int_0^T \Phi dx=0$, $A=\text{diag}\,(\varrho_1, \cdots, \varrho_n)$, ϱ_μ complex constants incongruent with each other mod ωi. The following method of successive approximations avoiding secular terms is then applied: Let

$$Y_0=\text{diag}(e^{\tau_1 x}, \cdots, e^{\tau_n x}),$$

the τ_μ being certain unknown constants; assuming Y_{m-1}, $m\geq 1$, already defined and belonging to the class C_ω [see same Riv. **5**, 63–81 (1954); MR **17**, 31], define D_m as the "mean value" of

$$Y_0^{-1}\Phi Y_{m-1} \text{ and } Y_m=Y_0+\lambda Y_0\int Y_0^{-1}(\Phi-D_m)Y_{m-1}dx,$$

where $\int$ represents the primitive matrix which belongs to C_ω. It is shown that the D_m are diagonal matrices and that, if $|\lambda|$ is small enough, $D_m\to D=\text{diag}(d_1, \cdots, d_n)$, $Y_m\to Y$, $d_\mu=d_\mu(\tau_1, \cdots, \tau_n; \lambda)$; if the τ_μ are solutions of the system $\tau_\mu-\lambda d_\mu=\varrho_\mu$ (these solutions always exist), Y is a solution of the matrix differential equation and the τ_μ are the characteristic exponents. The method is applied to systems $y''+Ay+\lambda\Phi y=0$, y n-vector, $A=\text{diag}(\sigma_1^2, \cdots, \sigma_n^2)$, σ_μ real and positive, λ real, Φ real L-integrable periodic of period $T=2\pi/\omega$, $\int_0^T \Phi dx=0$. Then, if Φ is either an even function of x or a symmetric matrix and if the σ_μ are incongruent with each other mod ω, the absolutely continuous solutions are bounded for $|\lambda|$ small enough. It is shown that this result can be extended to the case where A is a function of λ and the assumption $\int_0^T \Phi dx=0$ is not satisfied. *J. L. Massera* (Montevideo).

Gambill, Robert A. Stability criteria for linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma **5**, 169–181 (1954).

Let $\ddot{Y}+A\dot{Y}+\lambda\Phi Y=0$, where Y is a column vector, A, Φ real matrices, λ real parameter, $\Phi=\Phi(t)$ periodic in t of period $T=2\pi/\omega$, $\int_0^T \Phi dt=0$, the Fourier series of Φ being absolutely convergent, $A=\text{diag}(\sigma_1^2, \cdots, \sigma_n^2)$ $(\sigma_j>0)$, the σ_j being incongruent with each other mod ω. L. Cesari [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (6) **11**, 633–695 (1940); MR **3**, 41] proved that the solutions are bounded provided that Φ is either an even function or a symmetric matrix, if $|\lambda|$ is small enough. In the present paper it is proved that the same conclusion holds true in the following more general case: $\Phi=\Phi_0+\Psi$, where $\Phi_0=\text{diag}(\Phi_1, \cdots, \Phi_k)$, Ψ has vanishing elements on or above (on or below) Φ_0 and each of the Φ_j is either an even function or a symmetric matrix. Still more general cases are considered where the σ_j and Φ are functions of λ and the assumption $\int_0^T \Phi dt=0$ is omitted. Another result is: the solutions of

$$\ddot{Y}+AY+\lambda\Phi Y+\lambda\Phi^*\dot{Y}=0$$

are bounded for $|\lambda|$ sufficiently small if the assumptions of the first paragraph of this review are satisfied and furthermore Φ^* is periodic of period T and has an absolutely convergent Fourier series, Φ is even and Φ^* odd. *J. L. Massera* (Montevideo).

Hartman, Philip, and Wintner, Aurel. On the assignment of asymptotic values for the solutions of linear differential equations of second order. Amer. J. Math. **77**, 475–483 (1955).

The authors consider asymptotically constant solutions of $x''+g(t)x'+f(t)x=0$, where g, f are continuous for large positive t. If g is real, f possibly complex, then two sets of sufficient conditions for there to be precisely one solution with $x(\infty)=1$ are

(a) $$\int^\infty t|f(t)|dt<\infty,\ g(t)\leq 0,$$

(or more generally $\int^\infty \max(g(t), 0)dt<\infty$), and

(b) $$\int^\infty |f(t)|dt<\infty,\ g(t)\leq \text{const.}<0.$$

In order that there be more than one solution with $x(\infty)=1$ it is sufficient that $\int^\infty |f(t)|dt<\infty$, $\text{Re}\, g(t)\geq$ const. >0, where both f, g may be complex. The proofs are by transformation to self-adjoint forms. Cases of failure of weaker conditions are noted. A third result deals with $x''+f(t)x=0$, where $\text{Re}\, f(t)\leq 0$; in particular, $\lim_{t\to\infty}|x(t)|$ exists for essentially one solution (apart from a constant factor). An appendix deals with nonoscillatory cases of $x''+f(t)x=0$, where f is real; there is essentially one "principal solution" such that $1/x$ is not of L^2 at ∞. A postscript gives a result to the converse effect. *F. V. Atkinson* (Canberra).

Markus, Lawrence. Continuous matrices and the stability of differential systems. Math. Z. **62**, 310–319 (1955).

The author presents a certain clarification of the relation between the c. e.'s .(characteristic exponents) and problems of stability for differential systems. Denote by $\mathfrak{M}_n$ the family of all n^2 matrices, whose elements are complex-valued functions of real t (time), continuous and bounded for $0\leq t<\infty$. A (in $\mathfrak{M}_n$) is said to be kinematically similar to B (in $\mathfrak{M}_n$), $A\sim B$, if there exists a P, such that P, P^{-1}, $\dot{P}$ are in $\mathfrak{M}_n$ and $-P^{-1}[\dot{P}-AP]=B$ $(0\leq t<\infty)$. Wide use is made of this kind of similarity. The work relates to the well known earlier works on stability and c.e.'s, due to Liapounoff and O. Perron, and to some fairly recent papers, like that of S. Diliberto [Contributions to the theory of nonlinear oscillations, Princeton 1950, pp. 1–38; MR **11**, 665]. Some estimates for c.e.'s are given; for example: if A (in $\mathfrak{M}_n$) is normal and $M(t)$ [$m(t)$] is the max [min] of the real parts of the characteristic roots of $A(t)$, then

$$\limsup t^{-1}\int_0^t m(s)ds\leq\lambda_i\leq\limsup t^{-1}\int_0^t M(s)ds,$$

the λ_i being the c.e.'s of A.

W. J. Trjitzinsky (Urbana, Ill.).

Bylov, B. F. On stability beyond the greatest characteristic exponent. Dokl. Akad. Nauk SSSR (N.S.) **103**, 181–184 (1955). (Russian)

Non-linear differential systems

$$(1)\qquad dx_i/dt=p_{ii}(t)x_i+F_i(t,x),\ x=(x_1,\cdots,x_n)\qquad(i=1,2,\cdots,n)$$

are considered with

$$(2)\quad F_i(t,0)=0,|F_i(t,x)-F_i(t,x')|\leq\delta|x-x'|\quad(i=1,\cdots n).$$

The linear system

$$(3)\qquad dx_i/dt=p_{ii}(t)x_i$$

is supposed to be regular in the sense of Lyapunov and the functions $p_{ii}(t)$ bounded with $\Lambda=\max\lambda_i$,

$$\lambda_i=\lim t^{-1}\int_0^t p_{ii}(\xi)d\xi$$

as $t\to+\infty$. System (3) is said to be stable beyond the greatest characteristic value if for every $\varepsilon>0$ there exists $\delta>0$ such that for every F_i satisfying (2) the characteristic exponents of the solutions of systems (1) (in the sense of Lyapunov) do not exceed $\Lambda+\varepsilon$. Then it is proved that this type of stability occurs if and only if, for every α and continuous bounded $g(t)$ with $t^{-1}\int_0^t g(\xi)d\xi\to\Lambda$ as $t\to+\infty$, there exists a constant $C=C(\alpha)$ and a continuous bounded $f_\alpha(t)$ with $\limsup t^{-1}\int_0^t f(\xi)d\xi\leq\alpha$ as $t\to+\infty$, such that $A\leq BC$, where $A=\exp\int_0^t[p_{ii}-g]d\xi$,

$$B=\exp\left[\alpha(t-\tau)+\int_\tau^t f_\alpha d\xi\right],$$

for all i and $t\geq\tau\geq 0$.

L. Cesari (Lafayette, Ind.).

Čečik, V. A. Necessary and sufficient conditions of semistability of a limit cycle. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 183–187 (1955). (Russian)

Let $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ be a second-order system, P, Q being twice continuously differentiable, $f=P/Q$. P. N. Papuš [Uspehi Mat. Nauk (N.S.) **7**, no. 4(50), 165–168 (1952); MR **14**, 472] considered necessary conditions in order that a closed trajectory C be a semistable "regular" limit cycle. In the present paper general conditions for a semistable (not necessarily "regular") cycle are found: A necessary condition is

$$\int_C(f_x-f_y)\cdot(1+f^2)^{-1}ds=0$$

(the case considered by Papuš was $f_x-f_y=0$); a sufficient condition is also considered which is rather complicated in the general case; in the particular case $f_x-f_y=0$ this sufficient condition is

$$\int_C(f_xf_y+f_{xy}f-f_{y^2})\cdot(1+f^2)^{-1}ds\neq 0.$$

J. L. Massera (Montevideo).

Urabe, Minoru. Infinitesimal deformation of cycles. J. Sci. Hiroshima Univ. Ser. A. **18**, 37–53 (1954); errata 401 (1955).

Let $\dot{x}=X(x,y,\varepsilon)$, $\dot{y}=Y(x,y,\varepsilon)$ be a second-order system, where it is assumed that X, Y are entire functions in x, y, analytic in ε for sufficiently small $|\varepsilon|$. The problem considered is the variation of the limit cycle when ε varies. If for $\varepsilon=0$ there is a limit cycle with period T such that $h_0=\int_0^T(X_x+Y_y)dt\neq 0$ ("roughly" stable or unstable), it is well known that, if $|\varepsilon|$ is small enough, there is one and only one limit cycle possessing the same type of stability. The author discusses the nontrivial cases where $h_0=0$ and even the case where there is a continuum of cycles, generalizing previous results of T. Uno [Proc. 1st Japan Nat. Congress Appl. Mech., Nat. Comm. Theoret. Appl. Mech., Sci. Council Japan, 1952, pp. 513–516; Math. Japon. **2**, 119–126 (1952); MR **14**, 377], G. F. D. Duff [Ann. of Math. **57**, 15–31 (1953); MR **14**, 751] and Krylov-Bogolyubov [Introduction to non-linear mechanics, Princeton, 1943; MR **4**, 142]. The precise statements are obviously too complicated to be reproduced here. *J. L. Massera* (Montevideo).

Gorbunov, A. D. On conditions of asymptotic stability of the zero solution of a system of ordinary linear homogeneous differential equations. Vestnik Moskov. Univ. **8**, no. 9, 49–55 (1953). (Russian)

The present paper preceeds one reviewed earlier [Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 39–78 (1954); MR **16**, 475] and is mentioned there, although it was not available to the reviewer at that time. The author proves that the condition for asymptotic stability in the sense of Lyapunov mentioned in the previous review is both necessary and sufficient. *L. Cesari* (Lafayette, Ind.).

Barrett, J. H. Behavior of solutions of second order self-adjoint differential equations. Proc. Amer. Math. Soc. **6**, 247–251 (1955).

The author applies the substitution $y(x)=\varrho(x)\sin\theta(x)$, $y'(x)=w(x)(r(x))^{-1}\varrho(x)\cos\theta(x)$, to the solutions of $(ry')'+qy=0$, where r, q are of class C and $r>0$, and derives results on the magnitude of y, y' and on the oscillation of solutions, in terms of the disposable function w. For example, a sufficient condition for y to be oscillatory is that

$$\lim_{x\to\infty}\int_a^x\{(wr^{-1}+qw^{-1})-|wr^{-1}-qw^{-1}|-|w'|w^{-1}\}dt=\infty,$$

for some positive w of class C'. Finally, taking $w=k=$ const., the asymptotic formula is derived for the case in which $\int_a^\infty|kr^{-1}-qk^{-1}|dt<\infty$. Comparison with known results is made. *F. V. Atkinson* (Canberra).

Lewis, D. C. Families of periodic solutions of systems having relatively invariant line integrals. Proc. Amer. Math. Soc. **6**, 181–185 (1955).

The author considers differential equations

(1) $$\dot{x}_i = X_i(x_1, \cdots, x_n) \quad (i=1, \cdots, n)$$

for which functions H and A_i of $x=(x_1, \cdots, x_n)$ exist satisfying

(2) $$\sum_{j=1}^{n} (\partial A_i/\partial x_j - \partial A_j/\partial x_i)X_j = \partial H/\partial x_i,$$

so that $\int_C \sum A_i dx_i$ is a relatively invariant integral and H is a first integral of (1). Let $x_i(t, c)$, where $c=(c_1, \cdots, c_m)$, be a family of solutions of (1) having period $T=T(c)$ and let $h(c)=H(x(t, c))$. It is then proved (under appropriate differentiability assumptions) that $\partial(h, T)/\partial(c_k, c_l)=0$ for all k, l. From this conclusions as to functional dependence of h, T are drawn. The theorem generalizes known results for Hamiltonian systems [see A. Wintner, The analytical foundations of celestial mechanics, Princeton, 1941, pp. 73–74; MR 3, 215]. *W. Kaplan* (Ann Arbor, Mich.).

Lewis, D. C. Periodic solutions of differential equations containing a parameter. Duke Math. J. **22**, 39–56 (1955).

Consider a system $\dot{x}=X(x, \mu, t)$ periodic in t with period 2π and having a solution $x=x_0(t)$ for $\mu=0$, periodic of period 2π. It is well known that under rather general assumptions the system then has periodic solutions for sufficiently small $|\mu| \neq 0$. The author finds bounds for the values of $|\mu|$ for which just one periodic solution exists in the neighborhood of $x_0(t)$; the computation of these bounds is too complicated to be reproduced here. In a certain sense these bounds are best possible. The method is based on the reduction by means of Green's functions to an integral equation which is solved by successive approximations. Both the real and complex cases are considered. *J. L. Massera* (Montevideo).

Savinov, G. V. Eigenschwingungssysteme mit stark ausgeprägter Nichtlinearität. Vestnik Moskov. Univ. **8**, no. 6, 77–83 (1953). (Russian)

L'A. considère l'équation $x''+g(x)=\mu \cdot f(x, x', \mu)$ dont il suppose qu'elle admet une intégrale périodique qu'il cherche à déterminer formellement par la méthode de perturbation, en supposant μ assez petit. Il obtient ainsi des expressions de la fréquence, de l'amplitude, à partir des expressions correspondantes pour l'intégrale périodique $x_0(t)$ de l'équation $x''+g(x)=0$. Exemple de la détermination de ces quantités dans le cas où $g=(1+ax^2)x$, $f=(1-bx^2)x'$. *Ch. Blanc* (Zbl **51**, 321).

Reissig, Rolf. Zur Theorie der erzwungenen Schwingungen. Math. Nachr. **13**, 309–312 (1955).

The system

(1) $$\ddot{u}_k = f_k(u_1, \cdots, u_n; v_1, \cdots, v_n; t), \quad k=1, \cdots, n,$$

$(v_j(t))=\dot{u}_j(t))$ of second-order differential equations is studied subject to the condition that f_k are periodic in t of period T and satisfy Lipschitz's condition everywhere in the phase space. Let $z(t)=(u_1(t), \cdots, u_n(t), v_1(t), \cdots, v_n(t))$ be a path in the $2n$-dimensional phase space defined by a solution $u_1(t), \cdots, u_n(t)$ of (1). Strengthening a stability notion due to Trefftz [Math. Ann. 95, 307–312 (1925)] the author calls a system (1) "extremely stable" if (i) each path $z(t)$ is bounded for $t>0$ and (ii) given any two paths $z_1(t)$ and $z_2(t)$, $z_1(t)-z_2(t) \to 0$ as $t \to +\infty$. It is shown that extreme stability is equivalent to the statement that the system has a periodic path (forced oscillation) to which all other solutions are asymptotic as $t \to +\infty$. The theorem is an immediate consequence of Trefftz's results [loc. cit.]. [Reviewer's comment: It is not essential that the dimension of the phase space be even; (1) could be replaced by a system of first-order equations.]

J. P. LaSalle (Notre Dame, Ind.).

Reissig, Rolf. Über eine nichtlineare Differentialgleichung 2. Ordnung. Math. Nachr. **13**, 313–318 (1955).

The differential equation is

(1) $$\ddot{x}+f(x)\dot{x}+x=e(t),$$

where $e(t)$ is continuous and periodic of least period L, and $f(x)$ is positive — except possibly at discrete points — and is piece-wise continuous. A suitable translation of x gives a forcing term whose average value is zero; it is therefore assumed that $\int_0^L e(t)dt=0$. By means of the author's result on "extreme stability" [see the paper reviewed above], two theorems by N. Levinson [J. Math. Phys. **22**, 181–187 (1943); MR **5**, 183] are extended and proofs are simplified. It is shown: if for some $a>0$, $\int_0^a f(\xi)\,d\xi=3M$ (or $\int_{-a}^0 f(\xi)\,d\xi=3M$), where $M=\max|\int_0^t e(\xi)\,d\xi|$, then (1) has a unique periodic solution; the solution is of period L and all other solutions approach it as $t \to +\infty$. The second theorem states that the same conclusion can be drawn if it is known that (1) has a solution which together with its first derivative is bounded for $t>0$. *J. P. LaSalle.*

Barbălat, I. Solutions bornées et solutions périodiques pour certaines équations différentielles non linéaires du second ordre. I. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 393–402 (1953). (Romanian. Russian and French summaries)

Barbălat, I. Solutions bornées et solutions périodiques pour certaines équations différentielles non linéaires du second ordre. II. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 503–515 (1953). (Romanian. Russian and French summaries)

Let $\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=e(t)$, e continuous and bounded for all t, f, g continuous for all $x, \dot{x}$. The following alternative assumptions on f, g are considered: (i) (first note) $g(x)/x \geq 1$, when $|x|>a>0$; $f(x, y)\cdot|y| \to \infty$ when $|y|>a$, $x^2+y^2 \to \infty$, $f(x, y)>-M$ when $|x|>a$, $|y| \leq a$, $xy>0$; (ii) (second note) $|g| \to \infty$ when $|x| \to \infty$, $xg>0$ when $|x|>a>0$; $f(x, y)\cdot|y|^\alpha>m>0$, $0 \leq \alpha<1$ when $|x|>a$, $|y|>a$, $f(x, y)\cdot|y|>-M$, $M>0$, when $|x|>a$, $|y|<a$, $xy>0$, $f(x, y)>-M$ when $|x| \leq a$, $|y|>a$. Then, if either (i) or (ii) is satisfied, there is a constant $k>0$ depending exclusively on f, g, e such that for any solution $|x|$ and $|\dot{x}|$ are ultimately bounded by k; if e is periodic and if the existence and uniqueness of Cauchy's problem are assured, there is at least one harmonic vibration.

J. L. Massera (Montevideo).

Halanay, A. Nouveaux critériums d'existence des solutions périodiques pour l'équation des oscillations non linéaires forcées. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 373–391 (1953). (Romanian. Russian and French summaries)

Let $\ddot{x}+f(x)\dot{x}+g(x)=e(t)$, f, g, e continuous, e periodic, $|e(t)| \leq E$. The equation admits a periodic solution if either (i) or (ii) is satisfied: (i) there are $x_0, x_1>x_0$, $k>0$ such that $g(x_0)+E=0$, $g(x)+E>0$ when $x>x_0$, $g(x)-E>k$ when $x>x_1$, $F_0(x)=\int_{x_0}^x f(x)dx$, $G_0(x)=\int_{x_0}^x [E+g(x)]\,dx$,

$F_0(x) \geq (8G_0(x))^{\frac{1}{2}}$ when $x_0 \leq x \leq x_2$, where $x_1 < x_2 < x_1 + L^2/k$, $L = 2M' + (E+M)(x_1 - x_0)/M'$, M, M' being the sup of $|g|$, $|F|$ in $[x_0, x_1]$; (ii) there are x_0, x_1 such that $g(x_0)+E=0$, $g(x_1)-E=0$, $-E<g(x)<E$ when $x_0<x<x_1$, $F_0(x) \geq (8G_0(x))^{\frac{1}{2}}$, $F_1(x) \geq (8G_1(x))^{\frac{1}{2}}$ in $[x_0, x_1]$, where $F_1(x) = \int_a^x f(x)dx$, $G_1(x) = \int_a^x [E-g(x)]dx$.

J. L. Massera (Montevideo).

Halanay, A. Relativement à la méthode du petit paramètre. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **6**, 483–488 (1954). (Romanian. Russian and French summaries)

This paper contains the following results. 1. If the second-order system $\dot{x} = X_0(x, y)$, $\dot{y} = Y_0(x, y)$ has a limit cycle, the system $\dot{x} = X_0(x, y) + \mu X_1(x, y, t, \mu)$, $\dot{y} = Y_0(x, y) + \mu Y_1(x, y, t, \mu)$, X_1, Y_1 periodic in t of period T, μ sufficiently small, has a periodic solution in the neighborhood of the limit cycle. 2. If the second-order system $\dot{x} = X_0(x, y, t)$, $\dot{y} = Y_0(x, y, t)$ has a periodic solution which is asymptotically stable in the large, then $\dot{x} = X_0 + \mu X_1(x, y, t, \mu)$, $\dot{y} = Y_0 + \mu Y_1(x, y, t, \mu)$ has a periodic solution for sufficiently small μ (X_0, Y_0, X_1, Y_1 periodic in t of period T). 3. If the system $\dot{x} = X_0(x, t)$, x being an n-vector, X_0 periodic in t of period T, has an asymptotically stable periodic solution $x = x_0(t)$ and if the system in the perturbations $y = x - x_0(t)$ is autonomous, then $\dot{x} = X_0 + \mu X_1(x, t, \mu)$ has a periodic solution for sufficiently small μ. Theorem 3 is true as stated; Theorem 2 is true if the periodic solution for $\mu = 0$ is a harmonic vibration (which was probably implicitly assumed by the author) or at least a subharmonic vibration, i.e. if its period is commensurable with T, otherwise it is false. Theorem 1 is true if the limit cycle is either asymptotically stable or unstable and if its period is commensurable with T (in particular, if X_1, Y_1 do not depend on t); if the cycle is simply stable or semistable not even an almost periodic solution exists in general.

J. L. Massera (Montevideo).

Corduneanu, C. Solutions presque-périodiques des équations différentielles non linéaires du second ordre. Com. Acad. R. P. Romîne **5**, 793–797 (1955). (Romanian. Russian and French summaries)

Let $f(u, v, w)$ be real-valued and almost-periodic in u, uniformly in a finite (v, w)-region, let $\partial f/\partial v$ and $\partial f/\partial w$ exist, $\partial f/\partial v$ having a positive lower bound. A simple argument then shows that bounded solutions of $y'' = f(x, y, y')$, if any, must be almost-periodic; it appears that the boundedness of y' is also to be postulated, unless the hypotheses for $f(u, v, w)$ hold for all w.

F. V. Atkinson (Canberra).

Mitropol'skiĭ, Yu. A. On unsteady oscillations in systems with many degrees of freedom. Ukrain. Mat. Ž. **6**, 176–189 (1954). (Russian)

The author treats more general systems (including the case where gyroscopic terms appear) than in an earlier paper [Prikl. Mat. Meh. **14**, 139–170 (1950); MR **12**, 181]. An example of a Lagrangian system with two degrees of freedom is worked out in detail. If $\tau = \varepsilon t$ and $\dot{q} = dq/dt$, the type of Lagrangian under consideration includes

$$L = \tfrac{1}{2}\{\sum_{i,j=1}^{N} a_{ij}(\tau)\dot{q}_i\dot{q}_j + 2\sum_{i,j=1}^{N} g_{ij}(\tau)q_i\dot{q}_j - \sum_{i,j=1}^{N} b_{ij}(\tau)q_iq_j\} + \varepsilon L_1(\tau, q_i, \dot{q}_i).$$

N. Levinson (Cambridge, Mass.).

Aymerich, Giuseppe. Oscillazioni periodiche di un sistema di Rocard a due gradi di libertà. Rend. Sem. Fac. Sci. Univ. Cagliari **24**, 51–62 (1954).

Consider the system

$$\dot{x}_1 = y_1 - 2\varepsilon\alpha_1(x_1 - \tfrac{1}{4}\pi a_0 \operatorname{sgn} x_1),\ \dot{y}_1 = \omega_1^2 x_1 + k_2 x_2,$$

$$\dot{x}_2 = y_2 - 2\varepsilon\alpha_2 x_2,\ \dot{y}_2 = -\omega_2^2 x_2 - k_2 x_1,$$

where α_1, α_2, a_0, ω_1, ω_2 are positive constants, k_1, k_2 are constants having the same sign, $k_1k_2 - \omega_1^2\omega_2^2 > 0$ and ε is a small parameter. The existence and stability of periodic solutions is discussed by the methods of Krylov and Bogolyubov [Introduction to non-linear mechanics, Princeton, 1943; MR **4**, 142]; jump (hysteresis) phenomena are observed.

J. L. Massera (Montevideo).

Haacke, Wolfhart. Über die nichtlineare Mechanik. Phys. Bl. **9**, 398–405 (1953).

Expository paper.

Sawaragi, Yoshikazu, and Hirosawa, Eiichi. On the forced steady oscillations in a nonlinear closed loop control system. Mem. Fac. Engrg. Kyoto Univ. **17**, 166–174 (1955).

Response curves are obtained for the harmonic and combination oscillations of a control circuit represented by the equation $\dddot{\phi} + S\ddot{\phi} + \dot{\phi} + RS\phi - TS\phi^3 = s\dot{\sigma}$ where R, S, T, s are constants and $\sigma = \sigma_0 \sin \mu\tau$. The stability of the harmonic oscillations is discussed in some detail following the method of Andronow and Witt [see, e.g., J. J. Stoker, Non-linear vibrations in mechanical and electrical systems, Interscience, New York, 1950, pp. 153–163; MR **11**, 666].

H. A. Antosiewicz (Washington, D.C.).

Sansone, G., e Conti, R. Sull'equazione di T. Uno ed R. Yokomi. Ann. Mat. Pura Appl. (4) **38**, 205–212 (1955).

In a previous paper [same Ann. (4) **37**, 37–59 (1954); MR **16**, 478] the authors have studied periodic solutions of the system of equations

$$\dot{x} = y^3 - (x+1)[(x-1)^3 + \lambda],\ \dot{y} = -xy.$$

Among the results given in that paper is a certain uniqueness theorem concerning the periodic solutions. In the present paper the authors correct an error in the proof of the theorem, and give an alternative proof.

L. A. MacColl (New York, N.Y.).

Nagumo, Zin-iti. On a forced discontinuous oscillation. Proc. Fac. Engrg. Keio Univ. **7** (1954), 36–43 (1955).

The author investigates subharmonic resonance of relaxation oscillations of the equation

$$\delta(x^2 - 1)dx/d\tau + x = b \sin(\tau + \phi),\ 1 < |x| \leq 2,\ b \text{ small}.$$

His procedure is to expand $\tau(x)$ as a series in b and determine the coefficients as functions of x. He traces the changes of the phase ϕ during the jumps and observes that these tend toward appropriate integral multiples of 2π only near odd-order subharmonics. Several oscillators governed by the above equation are shown. Comparison with one experiment is given.

E. Pinney (Berkeley, Calif.).

Fage, M. K. Reduction to simplest form of Cauchy's problem for an ordinary linear differential equation of second order. Dokl. Akad. Nauk SSSR (N.S.) **99**, 909–912 (1954). (Russian)

Consider the differential operator $L(y) = y'' + q(x)y$, q continuous in $[0, +\infty)$; L maps Φ into C where Φ is the

linear space of twice continuously differentiable functions y in $[0, +\infty)$ with $y(0)=y'(0)=0$ and C the space of continuous functions in $[0, +\infty)$. The inverse operator A is an integral operator with a kernel function $A(x, s)$, $x \geq s \geq 0$. The simplest case is $L^{(0)}=d^2/dx^2$, the kernel $A^{(0)}$ being $x-s$. The main result of this paper is: there is an invertible operator H of C onto C, $H(\Phi)=\Phi$, such that $A=HA^{(0)}H^{-1}$, $L=H(d^2/dx^2)H^{-1}$. *J. L. Massera.*

Davies, Robert. Expansions in series of non-orthogonal eigenfunctions. Indust. Math. **4**, 9–16 (1953).

The modified Sturm-Liouville problem, consisting of the Sturm-Liouville differential equation and two linear boundary conditions that involve values of the second ordered derivative of the function, is considered here. The modification of the inner product of two functions, involved in dealing with the expansion of arbitrary functions in series of the non-orthogonal characteristic functions of the problem, is noted. The characteristic functions become orthogonal with respect to the modified inner product, and the technique of writing the coefficients in the expansion is given. The problem of determining the longitudinal vibrations of a bar with a mass attached to one end is used as an illustration. References are cited for the analytical details. *R. V. Churchill.*

See also: von Sanden, p. 89; Eltermann, p. 89; Boček, p. 90; Artemov, p. 90; Albrecht, p. 90; Azbelev, p. 90;

Partial Differential Equations

Gheorghiu, Octavian Em. Sur un système d'équations fonctionnelles. I. Com. Acad. R. P. Romîne **4**, 573–578 (1954). (Romanian. Russian and French summaries)

The author has previously shown [Bul. Şti. Tehn. Politehn. Timişoara **1**, no. 4, p. 12ff. (1950) (unavailable for review)] that the solutions $T_j=T_j(u, v, w)$, $j=1, 2, 3$, of the system of equations in total differentials,

$$dT_1=(1-T_1^2)du+(T_3-T_1T_2)dv+(T_2-T_3T_1)dw,$$
$$dT_2=(T_3-T_1T_2)du+(1-T_2^2)dv+(T_1-T_2T_3)dw,$$
$$dT_3=(T_2-T_1T_3)du+(T_1-T_2T_3)dv+(1-T_3^2)dw,$$

are characterized by the system

$$T_j(u+u', v+v', w+w')=\frac{T_j+T_j'+T_kT_l'+T_lT_k'}{1+\sum_{m=1}^{3}T_mT_m'},$$

where $j, k, l=1, 2, 3$ in cyclic order and $T_j'=T_j(u', v', w')$. He now determines the general solution of the latter system; the three functions thus determined involve nine arbitrary constants. A more general system of four equations in five unknown functions also is solved, the solution involving one arbitrary function $\mu(u, v, w)$ and nine arbitrary constants. *E. F. Beckenbach.*

Gul', I. M. Cauchy's problem for some partial differential equations with functional arguments. Uspehi Mat. Nauk **10**, no. 2(64), 153–156 (1955). (Russian)

The author considers equations

(1) $\Phi[x_i, u(\alpha_i^k), u_{x_j}(x_i)]=0$ $(i, j=1, \cdots, n; k=1, \cdots, K)$,

where $u(\alpha_i^k)=u(\alpha_1^k, \cdots, \alpha_n^k)$, $\alpha_i^k=\alpha_i^k(x_1, \cdots, x_n)$ and the functions Φ, α_i^k are continuously differentiable up to order two in a certain domain. Under suitable additional hypotheses it is indicated how the problem of Cauchy for (1) can be solved by utilizing a suitably modified method of Cauchy (used by the latter for the classic problem and involving the use of characteristics). Similar indications are given for the second-order quasi-linear equation

$$\sum_{j=1}^{n} a_j[x_i, u(\alpha_i^k)]u_{x_j}(x_i)+b[x_i, u(\alpha_i^k)]=0.$$

W. J. Trjitzinsky (Urbana, Ill.).

Vigranenko, T. I. On Cauchy's problem for integro-differential equations in first-order partial derivatives. Uspehi Mat. Nauk **10**, no. 2(64), 147–152 (1955). (Russian)

Let D be a bounded region in n-space with coordinates (x_i) $(i=1, \cdots, n)$ and let C' denote the set of functions with continuous derivatives on D. For $M=(x_i) \in D$ and $N=(y_i) \in D$ and $u \in C'$ let

$$X[u]=\sum_{1}^{n} X_i(M)\partial u/\partial x_i,$$

$$Y[u]=\sum_{1}^{n} Y_i(N)\partial u/\partial y_i+Y_{n+1}(N)u(N),$$

with $X_i \in C'$, and Y_i and Y_{n+1} integrable. Let $K(M, N)$ and its first partial derivatives be bounded. The problem is to find an integral surface of

(1) $$X[u]=\lambda \int_D K(M, N)Y[u]dN$$

containing a given non-characteristic $(n-1)$-manifold

(2) $$x_i=\varphi_i(t_\alpha),\ u=\varphi(t_\alpha)\ (\alpha=1, \cdots, n-1).$$

It is assumed that the solutions of the system (S) $dx_i/ds=X_i(M)$ with initial values $x_i=\varphi_i$ cover D in such a way that there exist ψ, ψ_α in C' with (*) $s=\psi(x_i)$, $t_\alpha=\psi_\alpha(x_i)$. By integrating (1) along a solution of S and substituting from (*) the author arrives at an integral equation (9) $u(M)=\theta(M)+\lambda \int_D P(M, N)Y[u]dN$ equivalent to the given problem; here $\theta(M)=\varphi(\psi_\alpha)$ and $P(M, N)=\int_0^s K(x_i(s, t_\alpha), N)ds$ with s, t_α replaced by ψ, ψ_α. It is proved as a lemma that $X[P]=K$. An application of the operator Y to (9) yields

(12) $$v(M)=\Phi(M)+\lambda \int_D Q(M, N)v(N)dN,$$

with $\Phi=Y[\varphi]$ and $Q=Y[P]$. Theorem 1: If λ is not an eigenvalue of Q and v_0 is the corresponding solution of (12), then (9) has a unique solution given by (13) $u(M)=\theta(M)+\lambda \int_D P(M, N)v_0(N)dN$. Next let $\varphi=0$, let λ' be an eigenvalue of Q of rank r and let $v_\varrho(M)$ $(\varrho=1, \cdots, r)$ be independent corresponding solutions of the homogeneous equation (12). Theorem 2: The functions (15) $u_\varrho(M)=\lambda' \int_D P(M, N)v_\varrho(N)dN$ satisfy (1) for $\lambda=\lambda'$ and vanish when $x_i=\varphi_i$, and give all independent functions with both these properties. Theorem 3: If $v_0(M)$ is a solution of (12) for $\lambda=\lambda'$, then for $\lambda=\lambda'$ (9) has the solution $u(M)=\theta(M)+\lambda' \int_D P(M, N)v_0(N)dN+\sum A_\varrho u_\varrho(M)$, where the A_ϱ are arbitrary constants and the u_ϱ are from (15). *F. A. Ficken* (Knoxville, Tenn.).

Kasuga, Takashi. On the homogeneous linear partial differential equation of the first order. Osaka Math. J. **7**, 39–67 (1955).

Let G be an open domain in the real $(n+1)$-space of coordinates $x, y_1, \cdots, y_n$, and f_μ $(\mu=1, 2, \cdots, n)$ functions defined and continuous in G with continuous derivatives $\partial f_\mu/\partial y_\lambda$. The author considers quasi-solutions in G of the linear homogeneous partial differential equation

(1) $$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x, y_1, \cdots, y_n)\frac{\partial z}{\partial y_\mu}=0,$$

i.e. continuous functions z which have derivatives with respect to x and the y_i in G with the possible exception of a denumerable set and which satisfy (1) almost everywhere in G. If z satisfies (1) everywhere in G and is totally differentiable in G, it is called an ordinary solution. The following theorems are proved. Theorem 1. A quasi-solution is constant on the characteristic curves, i.e., the curves satisfying:

$$\frac{dy_\lambda}{dx}=f_\lambda(x, y_1, \cdots, y_n) \quad (\lambda=1, 2, \cdots, n).$$

Theorem 2 states essentially that (under a certain additional assumption) the quasi-solution of the initial value problem for (1) with totally differentiable initial values is unique and that this unique quasi-solution is an ordinary solution. Theorem 3 states that for $n=1$ the notion of quasi-solution and ordinary solution coincide.

E. H. Rothe (Ann Arbor, Mich.).

Moisil, Gr. C. Sur les invariants des systèmes de Vecua. I. Com. Acad. R. P. Romîne **5**, 7–11 (1955). (Romanian. Russian and French summaries)

Moisil, Gr. C. Sur les invariants des systèmes de Vecua. III. Com. Acad. R. P. Romîne **5**, 13–19 (1955). (Romanian. Russian and French summaries)

Formal considerations concerning linear systems of two first-order partial differential equations.

L. Bers (New York, N.Y.).

★**Douglis, Avron. Function-theoretic properties of certain elliptic systems of first-order linear equations.** Lectures on functions of a complex variable, pp. 335–340. The University of Michigan Press, Ann Arbor, 1955. $10.00.

This is a brief exposition of the author's function theory for elliptic systems of first-order linear equations in $n>2$ independent variables. The emphasis is on the so-called generalized Beltrami equations which have many properties in common with the classical Beltrami equations but cannot, surprisingly enough, be transformed into equations with constant coefficients. A full presentation appeared elsewhere [Comm. Pure Appl. Math. **6**, 291–298 (1953); MR **16**, 257].

L. Bers (New York, N.Y.).

Oleĭnik, O. A. Boundary problems for partial differential equations with a small parameter in the highest derivatives and Cauchy's problem for nonlinear equations in the large. Uspehi Mat. Nauk (N.S.) **10**, no. 3(65), 229–234 (1955). (Russian)

Summary of the author's doctoral dissertation. Most of the results have been published earlier [Dokl. Akad. Nauk (N.S.) **79**, 735–737 (1951); **85**, 493–495 (1952); **95**, 451–454 (1954); Mat. Sb. N.S. **30(72)**, 695–702 (1952); **31(73)**, 104–117 (1952); Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 231–233 (1954); MR **13**, 559; **14**, 280, 560; **16**, 253].

Birman, M. Š. On Trefftz's variational method for the equation $\Delta^2 u=f$. Dokl. Akad. Nauk SSSR (N.S.) **101**, 201–204 (1955). (Russian)

The author gives a generalization of Trefftz's method for mixed self-adjoint boundary-value problems for $\Delta^2 u=f$. On parts of the boundary u and $\partial u/\partial n$ are given and on other parts combinations of the first three derivatives of u are prescribed. Such a problem can be written in the form

(1) $P_\Sigma U=0$, (2) $P_\Lambda U_n=BP_\Lambda U=0$.

U and U_n denote the following pairs of functions on the boundary:

$$U=(u, -\partial u/\partial n), \quad U_n=(\partial\Delta u/\partial n, \Delta u).$$

If S_1, S_2, L_1, L_2 are parts of the boundary Γ satisfying $S_1+L_1=S_2+L_2=\Gamma$, then $P_\Sigma U$ denotes the pair $(u|_{S_1}, -\partial u/\partial x|_{S_2})$, $P_\Lambda U$ denotes the pair $(u|_{L_1}, -\partial u/\partial n|_{L_2})$, and $P_\Lambda U_n$ is defined analogously. B is a self-adjoint operator on the boundary.

It is known that if

$$(3) \qquad Q(v, v)=\int_\Omega(\Delta v)^2 d\Omega+\int_\Gamma P_\Lambda V\cdot BP_\Lambda V ds$$

is non-negative, then the solution of the boundary-value problem minimizes the functional $A(v)=Q(v, v)-2\int_\Omega vf d\Omega$, where the functions v in comparison are to satisfy the essential boundary condition (1), while (2) comes out to be the natural boundary condition of the variational problem.

Consider an auxiliary purely natural boundary condition $U_n-AU=0$, where the self-adjoint operator A is chosen such that the functional $\bar{Q}(v, v)$ corresponding to (3) is non-negative and that the operator $M=B-P_\Lambda AP_\Lambda$ is positive definite. The following theorem is proved: The solution of the variational problem

$$\bar{Q}(v, v)+\int_\Gamma M^{-1}(P_\Lambda V_n-P_\Lambda AV)\cdot(P_\Lambda V_n-P_\Lambda AV)ds=\min,$$

where the functions v in comparison are to satisfy $\Delta^2 v=f$, coincides with the solution of the boundary-value problem (1), (2).

This result can be applied to the determination of approximate solutions of various thin-plate problems. The author mentions the following: 1) clamped plate, 2) freely supported plate, 3) free plate. The method seems to be appropriate also to cases where the plate is clamped or freely supported only on parts of the boundary.

W. Noll (Los Angeles, Calif.).

Keller, Joseph B., Lewis, Robert M., and Seckler, Bernard D. Asymptotic solution of some diffraction problems. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-81**, iii+79 pp. (1955).

Using the methods devised by Kline [Comm. Pure Appl. Math. **4**, 225–263 (1951); J. Rational Mech. Anal. **3**, 315–402 (1954); MR **12**, 886; **13**, 408; **15**, 433, 800] and Luneburg [Mathematical theory of optics, Brown Univ., 1944; Propagation of electromagnetic waves, New York Univ., 1948; MR **6**, 107], the authors give an extensive catalogue of the solutions of diffraction problems with excitation by plane, spherical and cylindrical waves in the limit $k\to\infty$.

A. E. Heins (Pittsburgh, Pa.).

Friedlander, F. G., and Keller, Joseph B. Asymptotic expansions of solutions of $(\nabla^2+k^2)u=0$. Comm. Pure Appl. Math. **8**, 387–394 (1955).

This is an effort to generalize the known direct methods of determining the behavior of the solutions of

$$(\nabla^2+k^2)u=0$$

for $k\to\infty$. (See for example the work of Kline and Luneburg cited in the preceding review.) Here the authors employ more general asymptotic forms which can be made to satisfy the above equation, but they do not prove that they are an expansion of some solution of this equation.

A. E. Heins (Pittsburgh, Pa.).

Payne, L. E. Inequalities for eigenvalues of membranes and plates. J. Rational Mech. Anal. **4**, 517–529 (1955).

Let D be a closed convex domain in the plane with boundary C. We consider the following two boundary-value problems:

$$\Delta u+\lambda u=0 \text{ on } D, \qquad u=0 \text{ on } C,$$
$$\Delta^2 v+\Lambda\Delta v=0 \text{ on } D, \qquad u=\partial u/\partial n=0 \text{ on } C.$$

The first problem is that of a membrane with fixed boundary, the second is the problem of buckling for a clamped plate. We denote the lowest eigenvalues by $\lambda_1, \lambda_2, \cdots$ and $\Lambda_1, \Lambda_2, \cdots$, respectively. The following remarkable result is proved which was conjectured by Weinstein: $\lambda_2 \leq \Lambda_1$. The equality sign holds only if D is of circular shape. *G. Szegö* (Stanford, Calif.).

Prodi, Giovanni. Sull'equivalenza tra la seconda formula di Green e la corrispondente equazione di Fredholm per l'equazione $\Delta_2 u+\lambda u=0$. Rend. Sem. Mat. Univ. Padova **24**, 103–122 (1955).

Let σ be a closed plane curve bounding an area τ. Then any solution of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda u=0 \tag{1}$$

defined in τ can, under certain conditions of regularity, be expressed by means of the first formula of Green, viz. for P inside σ,

$$u(P)=\frac{1}{2\pi}\int_\sigma\left\{u(M)\frac{\partial v(M,P)}{\partial \nu_M}-v(M,P)\frac{\partial u(M)}{\partial \nu_M}\right\}d\sigma_M,$$

where $v(M, P)$ is the fundamental solution of (1) and ν_M is the normal to σ at M drawn into τ. But if Q is outside σ,

$$0=\int_\sigma\left\{u(M)\frac{\partial v(M,Q)}{\partial \nu_M}-v(M,Q)\frac{\partial u(M)}{\partial \nu_M}\right\}d\sigma_M,$$

which the author calls the second formula of Green.

Now consider two functions $A(M)$, $B(M)$ integrable along σ such that, when Q is outside σ,

$$\int_\sigma\left\{A(M)\frac{\partial v(M,Q)}{\partial \nu_M}-B(M)v(M,Q)\right\}d\sigma_M=0. \tag{2}$$

Then if, when P is inside σ,

$$u(P)=\frac{1}{2\pi}\int_\sigma\left\{A(M)\frac{\partial v(M,P)}{\partial \nu_M}-B(M)v(M,P)\right\}d\sigma_M, \tag{3}$$

it was proved by Amerio [Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 79–102 (1945); MR 9, 37] that, for almost all M on σ,

$$u(P)\to A(M), \quad \partial u(P)/\partial \nu_M \to B(M) \tag{4}$$

as $P\to M$ along ν_M. But when M is on σ,

$$A(M)=\frac{1}{\pi}\int_\sigma\left\{A(N)\frac{\partial v(M,N)}{\partial \nu_N}-B(N)v(M,N)\right\}d\sigma_N \tag{5}$$

almost everywhere on σ. The present paper is concerned with finding necessary and sufficient conditions for the equivalence of (2) and (5). *E. T. Copson* (St. Andrews).

Sbrana, Francesco. Un nuovo procedimento per l'integrazione delle equazioni dell'elastodinamica e dell'elettromagnetismo. Rend. Sem. Mat. Univ. Padova **24**, 142–159 (1955).

Let Σ be a volume in three-dimensional space bounded by a closed surface σ; the unit vector normal to σ at Q, drawn out of Σ, is denoted by $\mathbf{n}(Q)$. Let $U(P, t)$ be a function of the rectangular cartesian coordinates (x, y, z) of the point P and of the time t. Let R be the distance from $P(x, y, z)$ to $Q(\xi, \eta, \zeta)$. Then if $\omega=c+i\lambda$, where c, λ are real, the transformation

$$u(P,t;\omega)=\iiint_\Sigma U(Q,t)\frac{e^{\omega R}}{R}d\xi d\eta d\zeta \tag{*}$$

has the inverse

$$\gamma U(P,t)=\lim_{\epsilon\to+0}\frac{i}{4\pi^2}\int_{\epsilon-\infty i}^{\epsilon+\infty i}\omega e^{\omega t}u(P,t;\omega)d\omega,$$

where $\gamma=0$ or 1 according as P lies outside or inside σ.

For brevity, the reviewer will denote (*) by

$$U(P,t)\to u(P,t;\omega).$$

This transformation has the following interesting properties:

$$\nabla^2 u(P,t;\omega)-\omega^2 u(P,t;\omega)=-4\pi\gamma U(P,t)$$

$$\nabla^2 U(P,t)\to\omega^2 u(P,t;\omega)-4\pi\gamma U(P,t)+\iint_\sigma\left[\frac{e^{\omega R}}{R}\frac{\partial U(Q,t)}{\partial n}-U(Q,t)\frac{\partial(e^{\omega R}/R)}{\partial n}\right]d\sigma$$

$$\operatorname{curl}\mathbf{S}(P,t)\to\operatorname{curl}\mathbf{s}(P,t;\omega)+\iint_\sigma\mathbf{n}(Q)\times\mathbf{S}(Q,t)\frac{e^{\omega R}}{R}d\sigma,$$

$$\operatorname{div}\mathbf{S}(P,t)\to\operatorname{div}\mathbf{s}(P,t;\omega)+\iint_\sigma\mathbf{n}(Q)\cdot\mathbf{S}(Q,t)\frac{e^{\omega R}}{R}d\sigma,$$

$$\operatorname{grad}\operatorname{div}\mathbf{S}(P,t)\to\operatorname{grad}\varphi(P,t;\omega)+\iint_\sigma\mathbf{n}(Q)\frac{e^{\omega R}}{R}\operatorname{div}\mathbf{S}(\varphi,t)d\sigma,$$

where φ is the transform of div $\mathbf{S}$.

These properties are used to solve (i) the equation of spherical waves, (ii) the equations of elastodynamics, (iii) Maxwell's equations. *E. T. Copson* (St. Andrews).

Jardetzky, W. S. On the general solution of a wave equation. Trans. New York Acad. Sci. (2) **15**, 297–301 (1953).

The author gives a considerable number of more or less general solutions of the wave equation

$$\nabla^2\varphi=\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} \quad (a \text{ a constant})$$

due to various scientists, without investigating if they satisfy Darboux's well known definition of the concept of general integral of a partial differential equation quoted in the beginning of the paper. No attempt is made to adapt the different solutions to given initial or boundary conditions. *H. Bremekamp* (Delft).

Castoldi, Luigi. Sulla più generale soluzione ondosa dell'equazione di D'Alembert. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 145–151 (1955).

The author obtains necessary and sufficient conditions on the amplitude function $A(P)$ and on the function $\phi(P)$ (which determines wave fronts or surfaces of constant phase) such that $\Psi_f(P,t)=A(P)\cdot f[\phi(P)-ct]$ shall satisfy the wave equation

$$\Delta\Psi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi=0$$

for arbitrary (unrestricted) $f\in C^2$. Here t is time, and P is a point of any n-dimensional space with positive definite Riemannian metric.

The condition on the surfaces of constant phase is that the orthogonal trajectories to these surfaces be geodesics. The determination of A is given explicitly in terms of the solution of a second-order elliptic partial differential equation on the $(n-1)$-dimensional surfaces of constant phase. [English-speaking readers might note that the somewhat unfamiliar notation of equation (8) corresponds to L. P. Eisenhart, Riemannian geometry, Princeton, 1949, p. 63, equation 21.1; MR 11, 687.] *R. B. Davis.*

Foiaş, Ciprian. Une condition d'unicité pour l'équation $\frac{\partial^2 z}{\partial x\,\partial y}=f(x, y, z)$. Com. Acad. R. P. Romîne 4, 565–572 (1954). (Romanian. Russian and French summaries)

The equation is $Dz=f(x, y, z)$ where

$$Dz=\lim\,[z(x+h, y+k)-z(x, y+k)-z(x+h, y)+z(x, y)](hk)^{-1} \text{ as } h, k\to 0$$

f is continuous and $|f(x, y, z)-f(x, y, z')|\leq K(x, y)\varphi(|z-z'|)$ with K summable, φ non-decreasing, and $\int_{0+}^{\infty} dt/\varphi(t)=\infty$ $(A>0)$. If $g(x)$ and $h(y)$ are continuous, with $g(x_0)=h(y_0)$, then the equation has at most one solution $z(x, y)$ with $z(x, y_0)=g(x)$ and $z(x_0, y)=h(y)$. *F. A. Ficken.*

Fourès-Bruhat, Yvonne. Solution élémentaire d'équations ultrahyperboliques. C. R. Acad. Sci. Paris 240, 395–396 (1955).

In this paper a fundamental solution of the ultrahyperbolic equation

$$\sum_{\alpha=1}^{n_1}\partial^2 u/(\partial x^\alpha)^2-\sum_{i=1}^{n_2}\partial^2 u/(\partial y^i)^2=0$$

is constructed by means previously applied by the author [Bull. Soc. Math. France 81, 225–288 (1953); MR 16, 43] to the wave equation in an even number of variables; these means are a development of a method originally due to S. Sobolev [Dokl. Akad. Nauk SSSR (N.S.) 1933, 258–262]. *A. Douglis* (New York, N.Y.).

Iacovache, Maria. Sur certaines intégrales curvilignes. Com. Acad. R. P. Romîne 1, 899–904 (1951). (Romanian. Russian and French summaries)

$\phi(x, y, z)$ being a solution of the system of partial differential equations $\phi_{xz}-\phi_{yz}=\phi_{yy}-\phi_{xz}=\phi_{zz}-\phi_{zy}=0$ (subscripts denote partial derivatives), the author determines polynomials U, V, W in the derivatives of ϕ for which $Udx+Vdy+Wdz$ is an exact differential as a consequence of the differential equations. *A. Erdélyi.*

Žitomirskiĭ, Ya. I. Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type. Mat. Sb. N.S. 36(78), 299–310 (1955). (Russian)

Soit S l'espace des fonctions sur R indéfiniment différentiables paires et à décroissance rapide [au sens de Schwartz, Théorie des distributions, t. II, Hermann, Paris, 1951; MR 12, 833]. Soit $j_p(x)=2^p x^{-p}\Gamma(p+1)J_p(x)$, J_p étant la fonction de Bessel usuelle. Pour $f\in S$, posons

$$\tilde{f}(s)=\int_0^\infty f(x)j_p(sx)x^{2p+1}dx,\quad p>0.$$

L'A. montre que $\tilde{f}$ est dans S (de sorte que $f\to\tilde{f}$ est un isomorphisme de S sur lui même). Le but de l'A. est ensuite de transposer ce résultat (méthode de Schwartz pour la définition de la transformation de Fourier des distributions tempérées); mais il utilise à cet effet une expression (T, φ) en général dénueé de sens. L'A. utilise ensuite l'application $f\to\tilde{f}$ (qui remplace la transformation de Fourier) à des opérateurs d'évolution du type $\partial/\partial t-P(B, t)$, $P(B, t)$=matrice à coefficients des polynomes en B, $B=\partial^2/\partial x^2+(2p+1)x^{-1}\partial/\partial x$. *J. L. Lions.*

Nicolescu, Miron. Sur l'équation de la chaleur. Com. Acad. R. P. Romîne 1, 747–751 (1951). (Romanian. Russian and French summaries)

Nicolescu, Miron. Une propriété caractéristique de moyenne des solutions régulières de l'équation de la chaleur. Com. Acad. R. P. Romîne 2, 677–679 (1952). (Romanian. Russian and French summaries)

Associate with the point $P(x, y, t)$ the paraboloid $(X-x)^2+(Y-y)^2+4(T-t)=0$. Let C and σ denote respectively the circle and the cap cut from this paraboloid by a plane parallel to the XOY plane. Let r be the radius of circle C and $\mu_0(u; r)$ be the mean value of the function u on the circle C. Further let $\mu_1(u; r)$ denote the mean value of u on σ. In the first paper listed above the author remarks that if $u=u(x, y, t)$ is an analytic function of its arguments, then $\mu_0(u, r)$ has the following representation:

$$\mu_0(u; r)=u(x, y, t)+\sum_{n=1}^{\infty} r^{2n}\sum_{p=0}^{n} a_n^{\,p}\frac{\partial^p}{\partial t^p}L^{n-p}u.$$

Here $Lu\equiv u_{xx}+u_{yy}-u_t$, $L^2u=L(Lu)$, $\cdots$, and the $a_n^{\,p}$ are certain constants. In the second paper it is shown that if $u=u(x, y, t)$ is a regular solution of $Lu=0$ in the neighborhood of the point P, then

$$\lim_{r\to 0}\frac{\mu_i(u, r)-u(x, y, t)}{r^2}=0\quad (i=0, 1).$$

F. G. Dressel (Durham, N.C.).

Nicolescu, Miron. Sur une propriété caractéristique de moyenne des fonctions polycaloriques. Com. Acad. R. P. Romîne 4, 551–554 (1954). (Romanian. Russian and French summaries)

A "mean-value" type of representation for solutions $u=u(x, t)$ of the equation $(\partial/\partial x^2-\partial/\partial t)^p u=0$ is written down. Here p is a positive integer, and the ranges for x and t are $-\infty<x<\infty$, $0<t<t_0$. The author gives proofs in another paper [Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 5, 243–332 (1954); MR 16, 709].

F. G. Dressel (Durham, N.C.).

See also: Volkov, p. 90; Korolyuk, p. 90; Artemov, p. 91; Teodorescu, p. 101.

Difference Equations, Special Functional Equations

Čermák, Jiří. Über lineare Systeme von Differenzengleichungen mit periodischen Koeffizienten. Časopis Pěst. Mat. 79, 141–150 (1954). (Czech. Russian and German summaries)

The author generalizes some results of T. Fort [Finite differences and difference equations in the real domain, Oxford, 1948; MR 9, 514] on the behavior of the solutions of homogeneous linear difference equations of the second order with periodic coefficients to systems of difference equations of the same type. Given the system

$$u_i(x+1)=\sum_1^n p_{ij}(x)u_j(x)\quad (i=1, \cdots, n)$$

with the p_{ij} periodic of period ω, he first observes that the solutions form an n-dimensional vector space over the complex numbers, with a basis called the fundamental system of solutions. If the fundamental system is represented by the matrix $U(x)$, then $U(x+\omega)$ is also a fundamental system, and there is a constant regular matrix A such that $U(x+\omega)=U(x)A$. This matrix is unique up to similarity. Several of his results are expressed in terms of "Weyr characteristics" whose definition would take up too much space. Two of those which are not so expressed are as follows: a necessary and sufficient condition that the above system should have nontrivial solutions of period ω is that the matrix A should have a characteristic root equal to unity; a sufficient solution that all the solutions belonging to a characteristic root a should approach zero as $x\to\infty$ is that the magnitude of a should be less than 1. He extends some of his results to systems of nonhomogeneous equations.

J. M. Danskin (Princeton, N.J.).

Èl'sgol'c, L. È. Stability of solutions of differential-difference equations. Uspehi Matem. Nauk (N.S.) **9**, no. 4(62), 95–112 (1954). (Russian)

An expository paper, discussing in addition to well-known material some results concerning the applicability of the second method of Lyapunov to differential-difference equations.

J. M. Danskin (Princeton, N.J.).

Gheorghiu, Oct. Em. Sur un système d'équations fonctionnelles. II. Com. Acad. R. P. Romîne **4**, 175–178 (1954). (Romanian. Russian and French summaries)

For the system of equations

$$\lambda(x,y,x',y')F(x+x',y+y')=F(x,y)F(x',y')+G(x,y)H(x',y')+G(x',y')H(x,y),$$
$$\lambda(x,y,x',y')G(x+x',y+y')=H(x,y)H(x',y')+G(x,y)F(x',y')+G(x',y')F(x,y),$$
$$\lambda(x,y,x',y')H(x+x',y+y')=G(x,y)G(x',y')+H(x,y)F(x',y')+H(x',y')F(x,y)$$

the author determines explicitly the totality of measurable solutions $F(x,y)$, $G(x,y)$, $H(x,y)$, $\lambda(x,y,x',y')$ in terms of an arbitrary measurable function $M(x,y)$ and of four arbitrary real constants b, b', c, c'.

E. F. Beckenbach (Los Angeles, Calif.).

Gheorghiu, Octavian Em. La solution générale mesurable pour un système d'équations fonctionnelles. Com. Acad. R. P. Romîne **2**, 199–203 (1952). (Romanian. Russian and French summaries)

For the system of equations

$$A(xy)=A(x)A(y)+B(y)C(x),$$
$$B(xy)=A(y)B(x)+B(y)D(x),$$
$$C(xy)=A(x)C(y)+C(x)D(y),$$
$$D(xy)=D(x)D(y)+B(x)C(y),$$

the author shows that any set of solutions must satisfy $B(x)=kC(x)$ and $A(x)-D(x)=\lambda C(x)$, where k and λ are constants. Making use of the known result that the only measurable solutions of the Cauchy equation $F(x+y)=F(x)+F(y)$ are the functions $F(x)=px$, where p is an arbitrary constant, the author determines the explicit form of all real and measurable solutions $A(x)$, $B(x)$, $C(x)$, $D(x)$ in the three cases $\lambda^2+4k>0$, $=0$, <0.

E. F. Beckenbach (Los Angeles, Calif.).

Ghermănescu, M. Equations fonctionnelles linéaires non homogènes. Com. Acad. R. P. Romîne **5**, 489–498 (1955). (Romanian. Russian and French summaries)

The author applies his method of "iterative constants" [Acad. Repub. Pop. Române. Bul. Şti. Şect. Şti. Mat. Fiz. **3**, 245–259 (1951); MR **15**, 713] to the solution of the nonhomogeneous equation

$$\sum_{i=0}^{n}\varphi^i(M)f(M_{n-i})=F(M),$$

where M has range in r-space, and $F(M)$, the $\varphi^i(M)$, and the operation $\theta(M)$ are known, with $M_1=\theta(M)$, $M_i=\theta(M_{i-1})$. Solutions are given in terms of an assumed particular solution and the previously discussed (loc. cit.) solution of the corresponding homogeneous equation. The method is extended to the solution of systems of nonhomogeneous linear functional equations.

E. F. Beckenbach (Los Angeles, Calif.).

Ghermănescu, M. Sur l'équation fonctionnelle

$$E(f)=\sum_{0}^{p}A_i f(x+\omega_i)=0. \quad \text{I.}$$

Com. Acad. R. P. Române **3**, 187–192 (1953). (Romanian. Russian and French summaries)

For the functional equation $\sum_{i=0}^{p}A_i f(x+\omega_i)=u$, $\omega_0=1$, where x is a real variable, the ω_i are positives or have positive real parts, and the A_i are constants satisfying k relations of the form $\sum_{i=0}^{p}A_i\omega_i^j=0$ $(j=0,\cdots,k-1)$, the author discusses solutions of several types. If z_r is a zero of multiplicity m of the characteristic equation

$$\theta(z)=\sum_{i=0}^{p}A_i e^{\omega_i z}=0,$$

then, for an arbitrary polynomial $P_m(x)$ of degree $m-1$ at most, the function $P_m(x)e^{z_r x}$ is a solution of the functional equation, as is any sum of such solutions. Again, any function of the form

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{e^{xz}\psi(z)}{\theta(z)}dz,$$

where $\psi(z)$ is analytic in $|z|\le 1$, is a special solution, and it is shown that such a special solution can be developed in a series of elementary solutions. *E. F. Beckenbach.*

Ghermănescu, M. Sur l'équation fonctionnelle

$$E_r(f)=\sum_{0}^{p}P_i(x)\,f(x+\omega_i)=0.$$

Com. Acad. R. P. Romîne **4**, 341–343 (1954). (Romanian. Russian and French summaries)

The analysis of the paper reviewed above is now extended to the more general equation $\sum_{i=0}^{p}P_i(x)f(x+\omega_i)=0$, where the $P_i(x)$ are polynomials of degree r at most. Solutions of the form $\int e^{xz}V(z)dz$ are investigated, where $V(z)$ is of class $C^{(r)}$; a resolvent ordinary differential equation of degree r at most, which $V(z)$ must satisfy, is determined. *E. F. Beckenbach* (Los Angeles, Calif.).

Ghermănescu, M. Sur l'équation fonctionnelle

$$E(f)=\sum_{0}^{p}A_i f(x+\omega_i)=0. \quad \text{II.}$$

Com. Acad. R. P. Romîne **5**, 499–502 (1955). (Romanian. Russian and French summaries)

The author uses the theory of residues to construct additional solutions of the equation appearing in the title of this paper. See the two preceding reviews.

E. F. Beckenbach (Los Angeles, Calif.).

Alaci, V. Fonctions presque homogènes. Com. Acad. R. P. Române 2, 113–115 (1952). (Romanian. Russian and French summaries)

The author considers functions $f(x, y, z)$ for which there exist functions $\Phi(t)$ and $\varphi_j(t)$, $j=1, 2, 3$, such that

$$f[x+\varphi_1(t),\ y+\varphi_2(t),\ z+\varphi_3(t)]=\Phi(t)f(x, y, z).$$

Several examples are given, including: $f=AK^{\alpha x+\beta y+\gamma z}$, with $\Phi=K^{\alpha\varphi_1+\beta\varphi_2+\gamma\varphi_3}$; and $f=AK^{\alpha x}+BK^{\beta y}+CK^{\gamma z}$, with $\varphi_1=\varphi/\alpha$, $\varphi_2=\varphi/\beta$, $\varphi_3=\varphi/\gamma$, and $\Phi=K^{\varphi}$. It is shown that if the functions involved are differentiable, then for such a function f we have

$$\varphi_1'(t)\frac{\partial f}{\partial x}+\varphi_2'(t)\frac{\partial f}{\partial y}+\varphi_3'(t)\frac{\partial f}{\partial z}=f(x, y, z)\frac{d}{dt}\log\Phi(t).$$

E. F. Beckenbach (Los Angeles, Calif.).

Integral Equations

Chang, Shih-Hsun. Integral equations with normal kernels. Acta Math. Sinica **4**, 1–20 (1954). (Chinese. English summary)

An English version of this paper appeared in Sci. Sinica 3, 369–385 (1954); MR **16**, 830.

Vasilache, Sergiu. Sur une équation intégrale du type Abel à deux variables. Com. Acad. R. P. Române **3**, 109–113 (1953). (Romanian. Russian and French summaries)

Formal generalization of Abel's integral equation to two variables. *A. Erdélyi* (Pasadena, Calif.).

Noble, B. On some dual integral equations. Quart. J. Math. Oxford Ser. (2) **6**, 81–87 (1955).

The author gives a formal solution of the dual integral equations

$$\int_0^\infty t^{\alpha}\varphi(t)J_\nu(\varrho t)dt=f(\varrho) \qquad (0<\varrho<1)$$

$$\int_0^\infty \varphi(t)J_\nu(\varrho t)dt=g(\varrho) \qquad (\varrho>1)$$

where f, g and α are given and φ is to be found.

A. E. Heins (Pittsburgh, Pa.).

Warzée, J. Les fonctions intégro-exponentielles et l'équation intégrale de l'assombrissement du centre au bord du Soleil. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 643–652 (1955).

In order to solve the integral equation

$$sI(s)=\int_0^\infty e^{-t/s}B(t)dt, \qquad I(1)=1,$$

the author sets

$$B(t)=a_0+a_1t-a_2E_2(t)-a_3E_3(t)-\cdots,$$

where $E_n(t)=\int_1^\infty x^{-n}e^{-tx}dx$, and obtains

$$I(s)=1-(1-s)a_1-a_2A_2(s)-a_3A_3(s)-\cdots,$$

where $A_2(s)$ and $A_3(s)$ are expressed in terms of elementary functions, and also tabulated. This method is said to yield good numerical results when $I(s)$ is given to about 3D.

A. Erdélyi (Pasadena, Calif.).

Fel'd, Ya. N. On infinite systems of linear algebraic equations connected with problems on semi-infinite periodic structures. Dokl. Akad. Nauk SSSR (N.S.) **102**, 257–260 (1955). (Russian)

The system referred to is (1) $\sum_{n=0}^{\infty}A_nZ_{m-n}=B_m$ $(m\geq 0)$, with $\{A_n\}$ as unknowns and the given quantities $\{Z_m\}$, $\{B_m\}$ subject to the relations (2) $r_1>r_2$, $r_0>r_2$ with $r_1^{-1}=\lim_{m\to\infty}|Z_m|^{1/m}$, $r_2=\lim_{m\to\infty}|Z_{-m}|^{1/m}$, $r_0^{-1}=\lim_{m\to\infty}|B_m|^{1/m}$. The author's solution of (1) is made to depend on the method of generating functions and on the non-homogeneous Hilbert boundary problem. [Reviewer's note: The limits in (2) can be replaced by lim sup.] Let $f(w)=\sum_0^\infty A_nw^n$, $g(w)=\sum_0^\infty B_nw^n$, $G(w)=\sum_{-\infty}^{\infty}Z_nw^n$, and let (c) be an arbitrary contour surrounding $w=0$ and lying in the ring $r_2<|w|<r=\min\{r_0, r_1\}$. Then system (1) reduces to finding $f(w)$ to satisfy the relations

$$(*) \qquad \int_{(c)}[f(w)G(w)-g(w)]w^{-m-1}dw=0 \quad (m\geq 0);$$

$$\int_{(c)}f(w)w^{-m-1}dw=0 \quad (m<0).$$

This leads to the Hilbert problem: find a function $\Phi(w)=\Phi^+(w)$ if $w\in S^+$, $=\Phi^-(w)$ if $w\in S^-$, where Φ^+ is holomorphic in S^+ (the exterior of (c)) and $\to 0$ as $w\to\infty$; Φ^- is holomorphic in S^- (the interior of (c)); and where the boundary condition $\Phi^+(t)=\Phi^-(t)G(t)-g(t)$ holds on (c). Known conditions for the existence of a solution are seen to hold, and the function $f(w)=\Phi^-(w)$ satisfies relations (*); hence the coefficients $\{A_n\}$ satisfy (1). The author illustrates the theory by solving a problem on waves. *I. M. Sheffer* (State College, Pa.).

Calculus of Variations

★ **Dedecker, Paul. Sur les intégrales multiples du calcul des variations.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 29–35. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author generalizes the work of certain authors, particularly E. Cartan [Leçons sur les invariants intégraux, Hermann, Paris, 1922] on the relation between Pfaff's problem and the calculus of variations. He associates with an exterior differential form Ω of degree $\mu<N$ in the space E of $(x^1, x^2, \cdots, x^N)$ the variational problem of finding the multiplicities V_μ which are extremals of the integral $\int_{V_\mu}\Omega$, and proves that the extremals will be given by N exterior differential equations of the form $\partial(d\Omega)/\partial(dx)$, where $d\Omega$ is the exterior differential of Ω.

The terminology of fibre spaces is used, but since the paper is concerned with local problems, the fibre spaces are topological products of a base space E and a fibre F. The author calls problem (L, E) that of finding the extremals of

$$\int L(t^\alpha, x^i, \partial x^i/\partial t^\alpha)dt^1\cdots dt^\mu \quad (\alpha=1, \cdots, \mu;\ i=1, \cdots, n)$$

in the space E of $t^1, \cdots, t^\mu, x^1, \cdots, x^n$. He associates with this a variational problem $(L, \mathcal{E})$ as follows: The space $\mathcal{E}$ is a fibre space with base E and fibre $(x_\alpha{}^i)$, the projection $\mathcal{E}\to E$ being defined by $(t^\alpha, x^i, x_\alpha{}^i)\to(t^\alpha, x^i)$. Given a manifold V_μ: $x^i=x^i(t^\alpha)$ of E, its image in $\mathcal{E}$ will be $x^i=x^i(t^\alpha)$, $x_\alpha{}^i=\partial x^i/\partial t^\alpha$. The images in $\mathcal{E}$ of manifolds in E are the integrals of the n Pfaffian equations $\omega^i=dx^i-x_\alpha{}^i dt^\alpha=0$. The problem $(L, \mathcal{E})$ is therefore a problem of restricted extrema, of finding among the solutions of $\omega^i=0$ the manifolds $x^i=x^i(t^\alpha)$, $x_\alpha{}^i=x_\alpha{}^i(t^\alpha)$ which are extremals of $\int Ldt$. The method of Lagrange multiples λ suggests a further modification of the problem into a problem $(L, \bar{\mathcal{E}})$ of finding the free extrema in a space $\bar{\mathcal{E}}$ of the variables t^α, x^i, $x_\alpha{}^i$, λ which is a new fibre space

both on $\mathcal{E}$ (fibre λ) and on E (fibre x_α^i, λ). The integrand function L is modified to a function Ω involving L, $\partial L/\partial x_\alpha^i$ and λ. The main theorem is deduced by applying his first theorem to the function Ω so formed.

The author proceeds to introduce generalizations of canonical variables, the Hamiltonian function and of the relative integral invariants of Cartan.

E. T. Davies (Southampton).

Šilova, G. I. Existence of an absolute minimum of multiple integrals of the calculus variations. Dokl. Akad. Nauk SSSR (N.S.) **102**, 699–702 (1955). (Russian)

The author considers the problem of minimizing

$$\int_{\bar{G}} F(x^1, \cdots, x^n, z, p^1, \cdots, p^n)dx^1 \cdots dx^n,$$

where $p^i = \partial z/\partial x^i$ and $\bar{G}$ is a closed bounded region of n-dimensional Euclidean space whose boundary G' is uniformly locally connected. A function z is in A^α if it is continuous, A.C. along almost all lines parallel to each of the coordinate axes, and the Dirichlet integral $\int_{\bar{G}} [\sum (p^i)^2]^{\alpha/2}$ is finite. The function F is subjected to continuity conditions and the condition

$$F(x, z, p) \geq m[\sum (p^i)^2]^{\alpha/2} \tag{1}$$

for all x and for some z and p (see below), m being a positive constant, as well as the condition that there exists a continuous convex function $\phi(p)$ such that $A\phi(p) \leq F(x, z, p) \leq B\phi(p)$ for all $x \in G$, all z with $|z| \leq M$, and all p, the A and B being constants, and the condition that the Weierstrassian be nonnegative. A function $z = f(x^1, \cdots, x^n)$ is admissible if it takes on preassigned continuous boundary values $\psi(x)$ on G', where $|\psi(x)| \leq M$, if it belongs to A^α, if $|z| \leq M$ on $\bar{G}$, and if it gives a finite value to the integral $\int F(x, z, p)dx$. The author proves that, for the case $\alpha = n > 2$, if the condition (1) holds for $|z| \leq M$ and sufficiently large values of $\sum (p^i)^2$, if there is an admissible function then there is a function which minimizes $\int F(x, y, p)dx$ in the class of admissible functions.

The case $\alpha > 2$, $n = 2$, was solved by Tonelli [Ann. Scuola Norm. Sup. Pisa (2) **2**, 89–130 (1933)]. The case of $\alpha = n = 2$ was solved essentially by Morrey [Univ. California Publ. Math. (N.S.) **1**, 1–130 (1943), see especially p. 45; MR **6**, 180] and later by Sigalov [Trudy Moskov. Mat. Obšč. **2**, 201–233 (1953); MR **15**, 442]. The methods used by the present author for the case $\alpha = n > 2$ are in large part adapted from those of Sigalov (loc. cit.). Details are not given.

J. M. Danskin (Princeton, N.J.).

Theory of Probability

★ **Rényi, Alfréd. Valószínűségszámítás. [The calculus of probabilities.]** Tankönyvkiadó, Budapest, 1954. 746 pp. 130 Ft.

Probability theory and its applications had been neglected in the curriculum of Hungarian universities until very recently when the author started to lecture regularly on these topics. He had therefore to prepare mimeographed notes for his students which lead, after repeated revisions, to the publication of the present book. It is thus the first modern Hungarian text book on probability theory and offers an excellent introduction into this field. The book is organized into three parts. The first part (chapters 1–5, pages 1–175) covers the elementary theory; it starts with a brief discussion of the Boolean algebra of events and adopts Kolmogorov's approach [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933] for the foundations of the theory. The second part (chapters 6–11, pages 176–439) is intended to guide the reader into the higher parts of the theory and it culminates in a discussion of the laws of large numbers. To indicate the scope of this part we mention that it includes Kolmogorov's necessary and sufficient condition for the validity of the strong law of large numbers as well as the law of the iterated logarithm for independently and identically distributed bounded random variables. The third part (chapters 12–16, pages 440–653) deals with a variety of more advanced topics such as characteristic functions and the central-limit theorem in its classical form. It also gives a brief introduction into several areas, for instance stochastic processes, in which active research is still going on at present.

The book contains two chapters on mathematical statistics: chapter 10 surveys briefly the elements, while chapter 15 deals with the theory of order statistics. The author presents in chapter 15 the approach to order statistics which the developed in a series of recent papers [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 467–503 (1953); Berichte über die Mathematiker-Tagung in Berlin, Deutscher Verlag der Wissenschaften, Berlin, 1953, pp. 203–212; Hajós and Rényi, Acta Math. Acad. Sci. Hungar. **5**, 1–6 (1954); MR **15**, 544, 885; **16**, 603; **15**, 972; **16**, 729]. The theorems of Kolmogorov and Smirnov on the limiting distribution of the maximum discrepancy between the theoretical and the empirical distribution function as well as some recent results of the author [loc. cit.] are given without proof, but the significance and also applications of these results are discussed in some detail. Fairly recent investigations of B. V. Gnedenko and V. S. Korolyuk [Dokl. Akad. Nauk SSSR (N.S.) **80**, 525–528 (1951); MR **13**, 570] dealt with the exact distribution of the maximum deviation (or absolute value of the maximum deviation) between two empirical distribution functions based on two samples of equal size drawn from the same population. These results are derived and the corresponding limiting distributions are determined.

The book is in several respects similar to B. V. Gnedenko's book [Course in the theory of probability, Gostehizdat, Moscow-Leningrad, 1950; MR **13**, 565]. Both books are text books rather than monographs, they are written at the same level and are very much aware of the needs of students. Moreover, the two authors have the same general outlook as evidenced also by occasional philosophical and political remarks and references to Marxist literature. There is however a big difference in the style of these two books. Unlike Gnedenko, the present author does not aim at a concise and terse presentation. He gives a detailed and interesting exposition and wishes to demonstrate that a mathematically sound foundation of the theory is important not only to the mathematician but also to the student of applied probability theory. The book is pedagogically very well organized; in this reviewer's opinion it will lead the student to the proper appreciation of the value of a rigorous mathematical argument even if he should be oriented primarily towards some specific applications. A valuable feature of the book is the large number of illustrative examples which are taken from a great variety of fields. These are discussed in detail and are dispersed throughout the text. They are

often used to give an additional motivation for the theoretical developments. At the end of each chapter there is a section with many problems. Some of these are just exercises while others are used to add supplementary material. A few problems contain even unpublished results. The book has four appendices. The first two provide tools from Boolean algebra, set theory and measure theory. Appendix 3 presents briefly the author's new axiomatic foundation for probability theory which was published recently [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 369–427 (1954); MR **16**, 599]. Appendix 4 contains a survey of the history of probability theory which was also published earlier [ibid. **4**, 447–466 (1954); MR **16**, 659]. It is regrettable that this book is useful only to Hungarian students; it would deserve to be added to the foreign language publications of the Hungarian Academy of Sciences. *E. Lukacs* (Washington, D.C.)

McFadden, J. A. Urn models of correlation and a comparison with the multivariate normal integral. Ann. Math. Statist. **26**, 478–489 (1955).

The problem considered first is as follows: Given n correlated random variables $x_1, x_2, \cdots, x_n$ where the marginal distribution of each x_i is such that the values $+1, -1$ are each assumed with probability $\frac{1}{2}$, then, in terms of the correlation coefficients r_{ij} between pairs (x_i, x_j), what is the probability that all n variables assume positive values? It is shown that this is equivalent to computing the probability that n successive draws in a generalized Polya urn scheme yield n balls of the same color. This idea leads to the desired formulae. The second problem considered is to find the probability that all n components of an n-dimensional random vector jointly distributed according to a standardized multivariate normal distribution, possess the same sign. It is shown that the two problems are related and that formulae in the multivariate case can be approximated by formulae obtained in the discrete case. It appears that the approximation is excellent if all the correlation coefficients are equal, but is not quite so good for the general case when they are unequal. *B. Epstein* (Stanford, Calif.).

Tumanyan, S. H. Asymptotic investigation of the multinomial probability distribution. Akad. Nauk Armyan. SSR. Dokl. **20**, 65–74 (1955). (Russian. Armenian summary)

Let $P_n(m_1, \cdots, m_s)$ be the probability of $m_1, \cdots, m_s$ occurrences, in independent n trials, of outcomes $A_1, \cdots, A_s$ (multinomial distribution). If the probability of A_i is p_i, the author supposes that p_i depends on n, $p_i = a_i/\phi(n)$ $(i \leq s-1)$. Here a_i is a positive constant (the reviewer does not understand why this constant is not taken to be 1) and $\phi(n)$ becomes infinite with n in such a way that $\phi_i(n)/n \to c_i$. Asymptotic expressions are found for P_n, with suitable restrictions on the m_j, when each c_i vanishes, when each c_i is positive and finite, when each c_i is infinite, and when these conditions are mixed. For example, in the first three cases, the asymptotic expressions are given by the local central limit theorem, a multinomial Poisson law, and by the limit relation $P_n(0, \cdots, 0, n) \to 1$.

J. L. Doob (Urbana, Ill.).

Poliščuk, E. M. On the mean value of a functional. Uspehi Mat. Nauk **10**, no. 2(64), 179–186 (1955). (Russian)

For each $t \in [0, 1]$ suppose that $a(t) < b(t)$, let $x(t)$ be a random variable uniformly distributed on the interval $[a(t), b(t)]$, and let the random variables of the family $\{x(t), 0 \leq t \leq 1\}$ be mutually independent. Certain functionals defined on the space of continuous functions on [0, 1], with values between $a(t)$ and $b(t)$ at each point t can be defined in terms of the values of the functions on a countable dense subset of [0, 1]. From this point of view, the functionals are then functions of the above random variables, are consequently themselves random variables, and their mean values can be defined as their expectations. It is shown that, for a simple class of functionals, the functional itself is, in an appropriate sense equal to its mean value with probability 1. The mean value is compared with the "Lebesgue integral" of the functional, defined like the mean value except that the measure of a subset of the interval $[a(t), b(t)]$ is now taken as the Lebesgue measure of the subset, instead of this measure divided by $b(t)-a(t)$. *J. L. Doob* (Urbana, Ill.).

Zoroa Terol, Procopio. Determination of the probability inside a region from its distribution function on the boundary. Rev. Acad. Ci. Madrid **48**, 151–169 (1954). (Spanish)

If $F(x, y)$ is a two-dimensional distribution function which has continuous partial derivatives F_x, F_y in the vicinity of a sufficiently regular closed curve C, the probability $P(S)$ which F assigns to the interior S of C is given by either of the counterclockwise line integrals $-\int F_x dx$, $\int F_y dy$ around C. Thus $P(S)$ is determined by the values of F in the vicinity of C but not, as shown by an example, by the values of F on C alone.

D. Blackwell (Berkeley, Calif.).

Dugué, D., et Girault, M. Fonctions convexes de Pólya. Publ. Inst. Statist. Univ. Paris **4**, 3–10 (1955).

Let $0 < \alpha \leq 1$. If $\varphi(x)$ is a real characteristic function convex for $x > 0$ such that $1 - \varphi(x) + x\alpha^{-1}\varphi_-'(x)$ increases from 0 to 1 as x varies from 0 to ∞, then a random variable with the characteristic function φ is the product of a random variable with characteristic function $1 - |t|^\alpha$ for $|t| \leq 1$ and 0 for $t > 1$ and another independent random variable. *K. L. Chung* (Syracuse, N.Y.).

Gilbert, Walter M. Projections of probability distributions. Acta Math. Acad. Sci. Hungar. **6**, 195–198 (1955). (Russian summary)

Given a two-dimensional Borel probability measure F, consider the projections $F(p_\theta^{-1}(E))$ $(0 \leq \theta < \pi)$, p_θ being the geometrical projection on the line $x_2 = (\tan\theta)x_1$ and E a one-dimensional Borel set. H. Cramér and H. Wold [J. London Math. Soc. **11**, 290–294 (1946)] have shown that

$$\varphi(\mathbf{r}) = \int e^{i\mathbf{r}\cdot\mathbf{x}} F(d\mathbf{x}) = \varphi_\theta(t) = \int e^{its} F(p_\theta^{-1}(ds))$$

$$(\mathbf{r} = (t\cos\theta, t\sin\theta))$$

which has the immediate consequence that F is completely specified by the collection of all projections.

Here, it is shown that the moments of F can be recovered from the moments of each infinite collection of projections and that, as soon as $\varphi_\theta(t)$ is analytic in t for two values of θ, $\varphi(t\cos\theta, t\sin\theta) = \varphi(\mathbf{r})$ is an analytic function of two complex variables.

Reviewer's note: in l. 10 on p. 196, read $\theta \notin (\pi/4, 3\pi/4)$ for $\theta \in [\pi/4, 3\pi/4]$.

H. P. McKean, Jr. (Princeton, N.J.).

Katz, Leo. Probability of indecomposability of a random mapping function. Ann. Math. Statist. **26**, 512–517 (1955).

Consider a finite set of N points. A mapping is called indecomposable on this set if the minimal non-null invariant set under the mapping is the whole set. A single-valued mapping is said to be a random mapping if it assigns, independently, to each point P_i belonging to the set one of the image points P_j, $j=1, 2, \cdots, N$, with equal probability $1/N$. The paper gives a proof of the following theorem: The probability that a random mapping on N points is indecomposable is

$$[(N-1)!/N^N]\sum_{M=0}^{N-1} N^M/M!.$$

An immediate corollary for the case where no point may be mapped into itself is also given. *M. Muller.*

Dobrušin, R. L. Lemma on the limit of compound random functions. Uspehi Mat. Nauk **10**, no. 2(64), 157–159 (1955). (Russian)

Let $\{\zeta(t), t\geq 1\}$, $\{\tau_n, n\geq 1\}$ be mutually independent sequences of random variables. Here τ_n is integer-valued. It is supposed that $[\zeta(t)-at^\alpha]/bt^\beta$ and $(\tau_n-cn^\gamma)/dn^\delta$ have respective limiting non-degenerate distribution functions F and G, when $t\to\infty$ and $n\to\infty$. The normalizing constants are specified and satisfy the relations $\alpha>\beta>0$, $b\neq 0$, $\gamma>\delta>0$, $d\neq 0$. It is proved that there are then constants as indicated such that $[\zeta(\tau_n)-gn^k]/hn^a$ has a limiting non-degenerate distribution function H when $n\to\infty$. The new constants and H are evaluated in terms of the given constants, F, and G. The proof is obtained by an adroit use of the fact that, if a sequence $\{K_n, n\geq 1\}$ of distribution functions converges to a distribution function K, at the continuity points of the latter, there is a sequence of random variables, $\{x_n, n\geq 1\}$, converging in measure to the random variable x, such that x_n has distribution function K_n and x has distribution function K. Particular cases of the above asymptotic result have been proved by Robbins [Bull. Amer. Math. Soc. **54**, 1151–1161 (1948); MR **10**, 385]. *J. L. Doob* (Urbana, Ill.).

Blum, J. R. On the convergence of empiric distribution functions. Ann. Math. Statist. **26**, 527–529 (1955).

Let $\mathcal{A}_1$ be the class of sets A in k-space each of which possesses the following property. If $x=(x_1, \cdots, x_k)\in A$ and $y=(y_1, \cdots, y_k)$ is such that $y_i<x_i$ for $i=1, \cdots, k$, then $y\in A$. Let $\mathcal{A}_j$, $j=2, \cdots, 2^k$, be the 2^k-1 classes of sets which can be obtained by reversing, one at a time, the k inequalities occurring in the definition of $\mathcal{A}_1$. Let $\mathcal{A}=\cup_{j=1}^{2^k}\mathcal{A}_j$. Let μ be a probability measure on the Borel sets of k-dimensional Euclidean space E_k. Let $\{X_n\}$, $n=1, 2, \cdots$, be a sequence of k-dimensional independent random vectors, distributed according to μ. Define $\mu_n(A)$ to be the proportion of observations among $X_1, \cdots, X_n$ which fall in A. The author proves that

$$P\{\lim_{n\to\infty}\sup_{A\in\mathcal{A}}|\mu_n(A)-\mu(A)|=0\}=1.$$

J. Wolfowitz (Ithaca, N.Y.).

Cote, Louis J. On fluctuations of sums of random variables. Proc. Amer. Math. Soc. **6**, 135–144 (1955).

This paper extends (i) a theorem of Chung [Ann. of Math. (2) **51**, 697–706 (1950); MR **11**, 731] on the limiting distribution of T_n=the number of crossings of a horizontal line by the successive partial sums of a sequence of independent random variables, and (ii) results of Chung and Erdös [ibid. **48**, 1003–1013 (1947); MR **9**, 292] on the lower limits of such sums. The generality in (i) consists in dropping the assumption that the random variables are identically distributed and in considering the crossings by the partial sums of a sequence $\{c_n\}$ of numbers. ($c_n=c$ reduces to the case considered by Chung.) The results of Chung and Erdös referred to above are extended to partial sums of independent, non-identically distributed random variables and to sums of independent, equidistributed lattice random variables. *G. Kallianpur* (Calcutta).

Derman, C., and Robbins, H. The strong law of large numbers when the first moment does not exist. Proc. Nat. Acad. Sci. U.S.A. **41**, 586–587 (1955).

Let S_n be the nth partial sum of a sequence of independent random variables with the common distribution F. Let $0<\alpha<\beta<1$; if $1-F(x)>0(x^{-\alpha})$ as $x\to\infty$ and $\int_{-\infty}^{0}|x|^\beta dF(x)<\infty$, then S_n/n tends to ∞ with probability one. To prove this one need only observe that the sum of the positive summands is of a greater order of magnitude than that of the negative ones, as follows from well-known results. The Y_n and Z_n are inaptly defined so that (9) as it stands is incorrect. An easy remedy is to define Y_n to be X_n if $X_n>0$ and 0 otherwise, and $Z_n=X_n-Y_n$.

The authors do not know if the first, obviously crude, condition on F can be replaced by $\int_0^\infty x^\alpha dF(x)=\infty$. A counterexample is trivial. For $x\to\infty$ let $F(x)>|x|^{-\beta-\varepsilon}$ with small $\varepsilon>0$. For $x>0$ let F be discrete with fast increasing large gaps and even faster decreasing probabilities. E.g., let $0<\varepsilon<\delta<(\beta+\delta)^{-1}-1$ and the jumps be at $2^{n^{1/\alpha}}$ with probabilities 2^{-n^1}. Let $k_n=2^{n^{1/\alpha}}$. It may be left to the authors to verify that $S_{k_n}<-k_n^{1/(\beta+\delta)}$ for all sufficiently large n with probability one. *K. L. Chung.*

Weiss, Lionel. The stochastic convergence of a function of sample successive differences. Ann. Math. Statist. **26**, 532–536 (1955).

Let f be a probability density function on the real line (with respect to Lebesgue measure) satisfying certain regularity conditions [which, according to a communication from the author, are not necessary to the conclusion which follows]. Let $X_1, \cdots, X_n$ be independent chance variables with common density f, let $Y_1^{(n)}\leq\cdots\leq Y_n^{(n)}$ be the ordered $X_1, \cdots, X_n$, and let $Q_n(t)$ be the proportion of the $n-1$ differences $Y_{i+1}^{(n)}-Y_i^{(n)}$ which are $>t/(n-1)$. The author proves that $\sup_{t\geq 0}|Q_n(t)-Ee^{-tf(X_1)}|\to 0$ with probability 1 as $n\to\infty$. One application of this result yields a result of B. Sherman [same Ann. **21**, 339–361 (1950); MR **12**, 192]. *J. Kiefer* (Ithaca, N.Y.).

Dobrušin, R. L. Central limit theorem for nonstationary Markov chains. Dokl. Akad. Nauk SSSR (N.S.) **102**, 5–8 (1955). (Russian)

Let $P(x, A)$ define a transition probability function. That is, this function defines a probability measure on the measurable sets A of a measurable space Ω for fixed x in Ω, and a measurable function of x for fixed A. Let ϱ be the upper limit (in a typographical slip called lower limit in the paper) of numbers α such that there is a measure μ with $\mu(\Omega)=\alpha$ and $P(x, B)\geq\mu(B)$ for all x, B. The number ϱ is called the measure of ergodicity of the transition probability function. For each value of $n>1$ let $n-1$ transition probability functions be given, with minimum measure of ergodicity ϱ_n, determining, together with some initial probability distribution, an n-step Markov process, with basic measure space the n-fold direct product of Ω with itself. Let ξ_{in} be a function on the ith factor space,

with distribution function F_{in}. Define $\zeta_n=\sum_{i=1}^{n}\xi_{in}$, denote the variance of a random variable ζ by $D\zeta$, and define $\zeta_n'=(\zeta_n-M\zeta_n)/(D\zeta_n)^{\frac{1}{2}}$. The following theorems are stated. (1) Suppose that, for some positive constant C and all i, n, $|\xi_{in}|\leq C$, $D\xi_{in}\geq 1/C$. Then, if $\varrho_n n^{\frac{1}{2}}\to\infty$, the distribution of ζ_n' is asymptotically the normal distribution with mean 0 and variance 1. (2) Suppose that (*) $\inf_{i,n} D\xi_{in}>0$, $\sup_{i,n} D\xi_{in}<\infty$. Then, if ζ_n' has a limiting distribution, this distribution is infinitely divisible. (3) Under (*), if $\lim_{\tau\to\infty}\int_{|x|>\tau} x^2 dF_{in}(x)=0$ uniformly as n, i vary, and if $\inf_n \varrho_n>0$, then the conclusion of (1) holds. These results include many known ones as special cases. For example, in treating (1), Linnik [Izv. Akad. Nauk SSSR. Ser. Mat. **13**, 65–94 (1949); MR **10**, 552] imposed the hypothesis $\varrho_n n^{\frac{1}{2}+\varepsilon}\to\infty$, and Sapogov [Dokl. Akad. Nauk SSSR (N.S.) **58**, 1905–1908 (1947); MR **9**, 361] imposed the hypothesis $\varrho_n n^{1/3+\varepsilon}\to\infty$, for some positive ε. In Linnik's treatment the state space Ω contained only finitely many points, and in Sapogov's treatment the space was a bounded domain of a Euclidean space, with the transition probability functions determined by densities relative to Lebesgue measure. *J. L. Doob.*

Morgenthaler, George W. A central limit theorem for uniformly bounded orthonormal systems. Trans. Amer. Math. Soc. **79**, 281–311 (1955).

The author obtains the following extension of a result of Salem and Zygmund on lacunary series [Proc. Nat. Acad. Sci. U.S.A. **33**, 333–338 (1947); MR **9**, 181]. Let $\{\phi_n(x)\}$ be a uniformly bounded orthonormal system of real-valued functions on a finite interval $[a, b]$. Then there exists a subsequence $\{\phi_{n_k}(x)\}$ and a real-valued function $f(x)$ on $[a, b]$, $\int_a^b f(x)dx=1$, $0\leq f(x)\leq M^2$, M the uniform bound of $\{\phi_n(x)\}$, such that for any measurable set $E\subset[a, b]$ with $m(E)>0$ and any $\{a_k\}$ an arbitrary sequence of real numbers satisfying: (a) $A_n=(a_1^2+a_2^2\cdots+a_n^2)^{\frac{1}{2}}\to\infty$ as $N\to\infty$, (b) $a_N=o(A_N)$, the distribution functions

$$F_N(y:E)=m(E)^{-1}m(\{x\in E|\sum_1^N a_k\phi_{n_k}(x)/A_N\leq y\}\qquad(N=1, 2, \cdots),$$

converge to a limiting distribution $F(y:E)$ at each point of continuity of the latter, and the characteristic function of $F(y:E)$ is

$$\Phi(\lambda:E)=m(E)^{-1}\int_E e^{-\lambda^2 f(x)/2}dx.$$

It is also shown that for every non-negative bounded normalized $f(x)$ an orthonormal family can be found which yields this $f(x)$ when the theorem is applied. The above theorem is further extended to complex-valued orthonormal systems. Analogous techniques yield a theorem on the limiting distribution (Gaussian) of appropriate sums of subsequences of the sequence $\{f(nx)\}$ where f is bounded and periodic. This last is a type of result previously considered by Kac [Ann. of Math. (2) **47**, 33–49 (1946); MR **7**, 436]. *J. Blackman.*

Karlin, Samuel. On the renewal equation. Pacific J. Math. **5**, 229–257 (1955).

The author treats the general, two-sided, renewal equation with a discrete or continuous variable. Unlike the probability case, questions of the existence, boundedness and uniqueness arise. Under conditions similar to those used by Chung and Pollard [Proc. Amer. Math. Soc. **3**, 303–309 (1952); MR **14**, 61] and Chung and Wolfowitz [Ann. of Math. (2) **55**, 1–6 (1952); MR **13**, 475] in the probability case, the author applies the general Wiener Tauberian theorem and the properties of slowly oscillating functions for his analysis. His results overlap those of W. L. Smith [Proc. Roy. Soc. Edinburgh. Sect. A **64**, 9–48 (1954); MR **15**, 722]. Roughly speaking, Karlin treats broader questions under somewhat more restrictive assumptions: finiteness of the mean and non-singularity of the distribution. The last section entitled "Abstract renewal equation" considers the operator point of view. *K. L. Chung* (Syracuse, N.Y.).

Root, W. L., and Pitcher, T. S. On the Fourier series expansion of random functions. Ann. Math. Statist. **26**, 313–318 (1955).

Let $x(t)$ be a mean-square continuous stochastic process which is stationary in the wide sense. Then the authors show that there exists a pairwise orthogonal set of random variables $\{x_k: k=0, \pm 1, \pm 2, \cdots\}$ such that

$$E\{|x(t)-T^{-\frac{1}{2}}\sum_{-n}^{n} x_k \exp(2\pi ikt/T)|^2\}\to 0$$

on $(0, T)$ as $n\to\infty$ if and only if x is periodic with period T. And, whether x is periodic or not, it is true that almost every sample function is of integrable-square on $(-T, T)$ and therefore that each such function has a Fourier series expansion which converges strongly in $L_2(-T, T)$. The coefficients in this expansion are random variables, and the paper concludes with a study of their asymptotic behaviour as $T\to\infty$.

The reader should be warned that the authors have not entirely avoided verbal ambiguity, and that there are several misprints. The reviewer found Theorem 2 particularly difficult to understand and was unable, without altering the hypotheses, to complete the required 'easy verification'. Here, as elsewhere in the paper, the symbol 'l.i.m.' is used without explanation and, although the authors' intention can usually be divined from the context, this practice is regrettable. *D. A. Edwards.*

★ **Dedebant, G. On a new definition of random function and its ergodic theorem.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 281–297. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

If $x(t)$ is a weakly stationary stochastic process, $t\geq 0$, normalized by $Ex(t)=0$, $Ex^2(t)=1$, it is noted that, considering each $x(t)$ as a point in Hilbert space, the curve C in Hilbert space determined by the stochastic process lies on the unit sphere and the length of the part of C for $0\leq t\leq T$ is kT for some constant k. For the case in which C is a circle, the correlation function of the process is $\phi(t)=Ex(0)x(t)=\cos kt$; the ergodic theorem mentioned in the title is that for this case the time average $M(T)=T^{-1}\int_0^T x(t)dt$ has $EM^2(T)=2(1-\cos kT)/k^2T^2$, so that $M(T)$ not only approaches 0 in mean square as $T\to\infty$ but vanishes identically for T a multiple of $2\pi/k$.

D. Blackwell (Berkeley, Calif.).

Ionescu Tulcea, C. T. Un théorème ergodique. Com. Acad. R. P. Române **1**, 23–27 (1951). (Romanian. Russian and French summaries)

In a previous paper [Ann. of Math. (2) **52**, 140–147 (1950); MR **12**, 266], the authors found an asymptotic expression for T^n, where T is a bounded linear operator of a certain type on a Banach space, and this result was applied to an operator defined by $Tf=\int f[y(t, x)]P(t, dx)$,

under suitable conditions on P, y. In the present paper, under slightly stronger conditions, and under the added restriction that $P(t, \cdot)$, which is a probability measure for each t, satisfies the inequality $P(t, A) \geq \lambda m(A)$, for some positive constant λ and some probability measure m, it is shown that T^n converges to an operator F, and in fact that $\|T^n - F\| \leq M(1+\varepsilon)^n$, for some positive constants M, ε. [The function value $y(t, x)$ is now denoted by $\tau_x(t)$.] The point of the result is in its application to probability chains with complete connection (liaisons complètes).

J. L. Doob (Urbana, Ill.).

Ciucu, Gh. Chaînes à liaisons complètes du type (B). Com. Acad. R. P. Române **1**, 455–460 (1951). (Romanian. Russian and French summaries)

Let $P^1(c|x)$ define a non-negative function of the indicated variables, where c is a sequence of members in a specified countable set I of real numbers, x is in I, and $\sum_{x \in I} P(c|x) = 1$. This function is thought of as defining ("chain with complete connections") a transition probability from the sequence c of past values to the next value x. It is supposed that (*) $P^1(c|x) \geq \lambda P^1(c'|x)$, for all c, c', x, and some positive constant λ. It is also supposed that P^1 satisfies a regularity condition "type B" introduced by Doeblin and Fortet [Bull. Soc. Math. France **65**, 132–148 (1937)]. It is then proved that, if $P^n(c|x)$ is the probability, with the past c, that n steps later the chain has value x, then $\lim_{n \to \infty} P^n(c|x)$ exists, uniformly as c, x, vary, defining a function of x alone. Under (*) and the (stronger than type B) hypothesis that the chain is Markovian, the result is a special case of one due to Kolmogorov [Math. Ann. **104**, 415–458 (1931)], who did not assume a countable state space. *J. L. Doob.*

Ciucu, George. Chaînes à liaisons complètes à densité. Com. Acad. R. P. Romîne **4**, 345–349 (1954). (Romanian. Russian and French summaries)

The result of the preceding review involved discrete distributions. In the present paper, an ergodic theorem of Ionescu Tulcea [see the second preceding review] is used to prove the corresponding result for chains whose transition probabilities are given by densities. Sufficient conditions are found that the n-step probability density converges (Cesaro or directly, according to the presence or absence of certain characteristic values) to a limiting density independent of the initial conditions. *J. L. Doob.*

Derman, Cyrus. Some contributions to the theory of denumerable Markov chains. Trans. Amer. Math. Soc. **79**, 541–555 (1955).

1. Suppose that each of denumerably many particles shifts from state to state as prescribed by the transition matrix of a Markov chain, independently of the other particles. The transition matrix P is the same for all particles, and is irreducible and recurrent. Let A_n be the vector whose ith component is the number of particles in state i at time n. Let v be a vector with positive components satisfying $vP = v$. Such a vector always exists, according to a theorem of the author [Proc. Amer. Math. Soc. **5**, 332–334 (1954); MR **15**, 722]. Then, for any constant initial vector A_0, the stochastic process $\{A_n, n \geq 0\}$ is a Markov process with stationary transition probabilities. If A_0 is a random vector, whose components are mutually independent, with Poisson distributions, and if $E\{A_0\} = v$, the A_n process is stationary. This stationary property is analogous to one discovered by the reviewer [Stochastic processes, Wiley, New York, 1953, Ch. VIII; MR **15**, 445]. Under stated conditions on A_0 and restrictions on P, it is shown that the A_n process is asymptotically stationary. Transition probability limit theorems for Markov processes are applied to the A_n process.

2. Let $N_n(i)$ be the number of times, in the first n transitions of a Markov chain with countably many states, that the state i is occupied. Then it is known [P. Lévy, Ann. Sci. Ecole Norm Sup. (3) **68**, 327–381 (1951); MR **13**, 959] that $\lim_{n \to \infty} N_n(j)/N_n(i) = \pi_{ji}$ exists. It is shown that, for a specified constant c_{ij} and ϕ any non-decreasing unbounded function on the positive integers, the inequality

$$\left| \frac{N_n(j) - N_n(i)\pi_{ji}}{[N_n(i) + N_n(j)]^{\frac{1}{2}} \phi[N_n(i) + N_n(j)]} \right| > c_{ij}$$

is satisfied for infinitely many n with probability either 1 or 0, the first if and only if $\sum [\phi(n)/n] \exp[-\phi^2(n)/2] = \infty$.

3. If finitely many states of a Markov chain are specified, let Y_n be the state occupied at the nth time the system occupies one of the specified states. The Y_n process is then a Markov chain with finitely many states and stationary transition probabilities. It is shown that, if the original chain is irreducible and recurrent, the (unique) vector of stationary absolute probabilities of the Y_n process has components proportional to the corresponding components of the vector v defined above.

J. L. Doob (Urbana, Ill.).

Neveu, Jacques. Semi-groupes généralisés et processus de Markoff. C. R. Acad. Sci. Paris **240**, 1046–1047 (1955).

Let Φ_s, $s \geq 0$, be a semi-group of positivity-preserving linear operators on $C(E^n)$ to $C(E^n)$ such that $\Phi_s \cdot 1 \leq 1$ and $\Phi_0 = I$. The weak continuity in S of Φ_s implies the existence of the infinitesimal generator Ω of Φ_s. The author assumes that $\Omega\,(\exp(it\mu)) = \Psi(t, z)$ exists and is equi-continuous in t in a vicinity of $t = 0$. He then states that, when f is in C^2, $\Omega f(z)$ is given by

$$\Omega f(z) = -\alpha(z) f(z) + \gamma^i(z) \partial_i f(z) + 2^{-1} s^{ij}(z) \partial_{ij} f(z) + \int_{E^n} [f(\xi + z) - f(z) - (1 + |\xi|^2)^{-1} \xi^i \partial_i f(z)] \times (1 + |\xi|^2) |\xi|^{-2} d_\xi \Psi'(z, \xi)$$

with the addional conditions (i)–(iii): (i) $\alpha(z)$, $\gamma^i(z)$ and $s^{ij}(z) + (\xi^i \xi^j / |\xi|^2) d_\xi \Psi'(z, \xi)$ is continuous in z; (ii) the family of measures $\Psi'(z, \xi)$ is weakly continuous in z; (ii) $\alpha(z) \geq 0$ and $s^{ij}(z) \eta_i \eta_j \geq 0$. It is stated also that the result may be extended to the "generalized semi-group" $\Phi_s{}^u \Phi_u{}^t = \Phi_s{}^t$, $s < u < t$, through the semi-group defined by $\Phi_a f(s) = \Phi_s{}^{s+a} f(s+a)$. Here $f(s)$ denotes a $C(E^n)$-valued continuous function. *K. Yosida* (Tokyo).

Doob, J. L. Martingales and one-dimensional diffusion. Trans. Amer. Math. Soc. **78**, 168–208 (1955).

To avoid the use of the existence theorems of parabolic differential equations, the author uses the direct construction of the stationary Markoff processes $x(t) = x(t, \omega)$ due to K. Itô [Mem. Amer. Math. Soc. no. 4 (1951) MR **12**, 724]:

$$x(t) = x(0) + \int_0^t m(x(s))\,ds + \int_0^t \sigma(x(s))\,dy(s),$$

where $y(s)$ is a Brownian motion, $0 \leq \sigma(\xi)$ for ξ in the "state interval" I, and

$$|m(\xi_2) - m(\xi_1)| \leq K(I_0)|\xi_2 - \xi_1|,\ |\sigma(\xi_2) - \sigma(\xi_1)| \leq K(I_0)|\xi_2 - \xi_1$$

for ξ_i in the compact subset I_0 of I. The §§2, 3 and 4 are respectively devoted to discussions, generalizations, and the "optional stopping" of the "standard Itô process" of the type given above. The semi-group

$$(U_t f)(\xi)=E\{f(x(t))|x(0,\omega)=\xi\}$$

generated by an Itô process (with bounded diffusion coefficients m, σ) is discussed in §§ 6 and 7. It is proved that, in an appropriate function space, the infinitesimal generator of the semi-group is $D=(2^{-1}\sigma^2)d^2/d\xi^2+md/d\xi$. § 8 gives the basis for the application of the martingale theory to the Itô process $\{x(t), 0\leq t<\infty\}$ with a compact state interval I such that $\sigma>0$ on I. Theorem 8.3 states that, for a bounded continuous function g on I, the processes

$$\{g(x(t)),\ 0\leq t<\infty\} \qquad \text{(in case } Dg=0),$$
$$\{\exp(-\lambda\tau_t)g(x(t)),\ 0\leq t<\infty\} \qquad \text{(in case } Dg=\lambda g),$$
$$g(x(t))-\lambda\tau_t,\ 0\leq t<\infty \qquad \text{(in case } Dg=\lambda)$$

are martingales. Here $\tau_t=\min(\tau, t)$ and τ=the time of first meeting of the trajectory $x(t)$ with the boundary of I. The § 9 is devoted to the behaviour of the sample functions of Itô processes. For example, theorem 9.1 states that, if $\sigma>0$ on a compact state interval I, almost every trajectory reaches the boundary of I. More detailed discussions of the trajectories for the state intervals $[r_1, r_2]$ and $[r_1, r_2)$ are given together with the illustration of the end-points r_i as "entrance" or "natural" boundary according to Feller's terminology [Ann. of Math. (2) **55**, 468–519 (1952); MR **13**, 948]. *K. Yosida* (Tokyo).

Fréchet, Maurice. Courbes aléatoires. Publ. Inst. Statist. Univ. Paris **4**, 11–20 (1955).

Dans des recherches antérieures, l'auteur avait déterminé expérimentalement les formes que prenait sur un plan un fil noué jeté au hasard sur ce plan. Bien qu'il ait momentanément abandonné l'étude de ces courbes, il présente ici, pour 22 formes du fil projeté, les courbes représentant les coordonnées d'un point du fil en fonction de l'abscisse curviligne. *J. Bass* (Paris).

Takács, Lajos. On processes of happenings generated by means of a Poisson process. Acta Math. Acad. Sci. Hungar. **6**, 81–99 (1955). (Russian summary)

English version of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 525–541 (1954); MR **16**, 723.

Takács, Lajos. Investigation of waiting time problems by reduction to Markov processes. Acta Math. Acad. Sci. Hungar. **6**, 101–129 (1955). (Russian summary)

English version of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 543–570 (1954); MR **16**, 723.

Morse, Philip M. Stochastic properties of waiting lines. J. Operations Res. Soc. Amer. **3**, 255–261 (1955).

The time behaviour of waiting lines is examined by determining the correlation function and corresponding power spectrum (as defined in noise theory) for length of line. The technique, as illustrated in detail for a single server with exponentially distributed service time, Poisson arrivals and order of arrival service, is to determine transition probabilities and from these the correlation function. The former are written in the spectral form of Ledermann and Reuter [Philos. Trans. Roy. Soc. London. Ser. A. **246**, 321–369 (1954); MR **15**, 625], who however give a more concise expression. The final expressions for both correlation function and power spectrum leave one integration to be done, but are approximated by simple expressions. With L and Δ^2 the mean and variance of the number either being served or waiting in equilibrium conditions, and taking time in units of the average service time, the approximation for the correlation function is $\psi(t)=L^2+\Delta^2\exp(-t/\Delta^2)$. A "relaxation time" is defined as "the mean time required for the queue to go from its mean-square deviation away from L back to $(1/e)$ of this deviation" and is said to be given approximately by $\tau=2\Delta^2$. Since $\Delta^2=\varrho/(1-\varrho)^2$, this shows that relaxation time is more sensitive to saturation ($\varrho\to 1$) than is the mean length of line. Finally indications are given for extension of this to many-server systems with exponential service time. *J. Riordan.*

★ **Yaglom, A. M. Über die statistische Umkehrbarkeit der Brownschen Bewegung.** Abhandlungen aus der Sowjetischen Physik, Folge III, pp. 7–42. Verlag Kultur und Fortschritt, Berlin, 1953.

Translation of Mat. Sb. N.S. **24(66)**, 457–492 (1949); MR **11**, 41.

Soulé, J. L. Energie cinétique moyenne d'une particule en mouvement brownien dans un fluide réel. Publ. Inst. Statist. Univ. Paris **4**, 93–104 (1955).

La formule classique du coefficient de diffusion du mouvement brownien d'une particule sphérique repose sur des hypothèses auxquelles on peut faire diverses objections. En particulier, dans la démonstration donnée par Langevin, on suppose que la particule est soumise à une résistance moyenne obéissant à la loi de Stokes, et qu'il y a indépendance entre les probabilités d'état des molécules et de la particule. L'auteur reprend la démonstration de Langevin en remplaçant la formule de Stokes par celle de Boussinesq, qui tient compte du passé du fluide, et en faisant les hypothèses suivantes: on peut négliger dans les équations de l'hydrodynamique les termes d'inertie; on peut assimiler la vitesse et l'accélération de la particule à la vitesse et à l'accélération hydrodynamique du fluide au contact de la particule. Il montre que ces hypothèses sont d'autant plus valables que la particule est plus grosse. En résolvant par les transformations de Fourier l'équation du mouvement (équation intégro-différentielle linéaire à second membre aléatoire), il obtient la formule

$$\lim\frac{\overline{x^2}}{t}=\frac{m\overline{U^2}}{3\pi\eta r}f\left(\frac{\varrho}{\varrho_0}\right)$$

où $\overline{x^2}$ est le carré moyen du déplacement de la particule pendant le temps t, $m\overline{U^2}$ l'énergie cinétique moyenne de la particule, r son rayon, ϱ_0 sa masse spécifique, ϱ la masse spécifique du fluide, η le coefficient de viscosité. f est une fonction inférieure à 1 dont l'expression est donnée, et qui tend vers 1 lorsque ϱ/ϱ_0 tend vers 0, auquel cas la formule ci-dessus se réduit à la formule classique d'Einstein.

J. Bass (Paris).

Bloh, È. L., and Harkevič, A. A. Geometric representations in communication theory. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 6, 91–100 (1955). (Russian)

This is an expository paper for engineers, generally akin to Shannon's paper in Proc. I.R.E. **37**, 10–21 (1949) [MR **10**, 464]. Reference is made to work of V. A. Kotel'nikov in 1946, but no literature is listed. The so-called

"sampling theorem" [cf., e.g., Shannon, Bell System Tech. J. **27**, 623–656 (1948); MR **10**, 133], which is quoted to the effect that a signal limited to a bandwidth F is determined over an interval $(0, T)$ by its values at $2FT$ equidistant points, is credited to Kotel'nikov. Actually, the statement is true only in an appropriate asymptotic sense as $T \to \infty$. *E. Reich* (Santa Monica, Calif.).

Lundkvist, Karl. Analysis of general theory for telephone traffic. Ericsson Technics **11**, 3–32 (1955).

The "general theory for telephone traffic" is that of an earlier paper [Ericsson Technics **9**, 111–140 (1953); MR **15**, 724]. It is used here to give what the author believes to be a simplified version of Conny Palm's result [ibid. **4**, 39–58 (1938)] that for random arrivals in a simple trunk group with blocked calls dismissed and disregarded, the equilibrium probabilities for number of busy trunks are independent of the call holding time distribution. His version rests on the assumption that the arrival times of existing calls are uniformly and independently distributed over past time. A simplification without this assumption and with a finite number of callers, apparently unknown to the author, has been given by L. Kosten [PTT Bedrijf **2**, no. 2, 42–45 (1948)]. The same assumption is used in a treatment, not fully developed, of the same system with a general inter-arrival time distribution. Finally the equilibrium probabilities for a divided access system with an infinite number of second choice (overflow) trunks are examined; a system of recurrences is obtained for overflow probabilities.

J. Riordan (New York, N.Y.).

Hammersley, J. M. Storage problems. Math. Ann. **128**, 475–478 (1955).

A formula (inversion of characteristic function) for the probability that the (chance) demand on a warehouse exceeds its capacity during the (chance) period between deliveries to the warehouse is derived under certain simple assumptions. *J. Kiefer* (Ithaca, N.Y.).

Rényi, Alfréd, und Szentmártony, Tibor. Wahrscheinlichkeitstheoretische Bestimmung von Reserven in Maschinenteilen und Ausrüstungsgegenständen. Mat. Lapok **3**, 129–139 (1952). (Hungarian. Russian and German summaries)

The author considers the problem of determining an adequate stock of spare parts. He assumes two causes, breakage and natural wear and tear, which necessitate replacement. A mathematical model involving certain simplifying assumptions is constructed. The time needed for replacement is considered to be a random variable. It is then possible to determine in this model the size of the basic stock which will, with a preassigned probability, assure that the replacement arrives before the basic stock is exhausted. *E. Lukacs* (Washington, D.C.).

on order

★ **Borel, Emile, et Chéron, André. Théorie mathématique du bridge à la portée de tous. 2ème éd.** Gauthier-Villars, Paris, 1955. xviii+424 pp. 2200 francs.

This edition differs from the first [1940; MR **8**, 470] in correction of known errors and the addition of a new chapter (pp. 383–413): Applications pratiques de la formule de Bayes.

Kornfel'd, M. On the theory of errors. Dokl. Akad. Nauk SSSR (N.S.) **103**, 213–214 (1955). (Russian)

The author questions the validity of the use of the normal law as the law of errors of physical measurements. Instead he assumes tacitly that the distribution of measurements has a unique median (the "true" value) and gives an interval estimation for this value.

E. Lukacs (Washington, D.C.).

Mathematical Statistics

★ **Haight, F. A. Index to the distributions of mathematical statistics.** Auckland University College, Auckland, New Zealand, 1955. 51 pp. (mimeographed).

Distributions are classified under the following headings: 1) Normal distributions. 2) Type III distributions. 3) Binomial distributions. 4) Discrete distributions. 5) Distributions on (a, b). 6) Distributions on $(0, \infty)$. 7) Distributions on $(-\infty, \infty)$. 8) Miscellaneous univariate. 9) Miscellaneous bivariate. 10) Miscellaneous multivariate. A very brief remark indicates something of the character of a paper treating a particular distribution and a code indicates the source.

Castoldi, Luigi. Formule ricorrenti per il calcolo dei cumulanti e dei momenti di una distribuzione statistica a partire dai corrispondenti momenti fattoriali. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 157–164 (1955).

Expository paper on relations between central moments, factorial moments and cumulants. *E. Lukacs.*

Castoldi, Luigi. „Densità media" di una distribuzione statistica e suo legame col „contenuto informativo" di N. Wiener. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 232–236 (1955).

The notion of mean density of a statistical distribution is defined. Wiener's [Cybernetics, Wiley, New York, 1948; MR **9**, 598] measure of the amount of information associated with the probability density is equal to the logarithm of the mean density. *S. Kullback.*

Marinescu, G. La fonction de distribution du maximum du module de n variables statistiques. Com. Acad. R. P. Române **1**, 309–313 (1951). (Romanian. Russian and French summaries)

Let $X_1, X_2, \cdots, X_n$ be n independent random variables and denote by $X = \max(|X_1|, |X_2|, \cdots, |X_n|)$. The author discusses the distribution and the expected value of X in certain simple particular cases. *E. Lukacs.*

Berz, F. On the cumulative effect of chance deviations. J. Roy. Statist. Soc. Ser. B. **16**, 269–284 (1954).

The author derives a majorant for the difference between the cumulative distribution function of a sum of rectangular variates and the Gaussian c.d.f. having the same mean and variance. *P. Whittle* (Wellington).

Durbin, J. Errors in variables. Rev. Inst. Internat. Statist. **22**, 23–32 (1954).

The author discusses several methods of fitting a linear relation to a set of observed variables from the viewpoint of the realism, in economics, of the assumptions they require. *J. Wolfowitz* (Ithaca, N.Y.).

Fisz, M. The construction of artificial populations and their application. Prace Mat. **1**, 174–182 (1955). (Polish. Russian and English summaries)

An expository paper, dealing with the use of random numbers for simulating samples from a population with preassigned distribution. *Z. Birnbaum.*

Okamoto, Masashi. Fit of a Poisson distribution by the index of dispersion. Osaka Math. J. **7**, 7–13 (1955).

The author considers a sample $x_1, x_2, \cdots, x_n$ from a Poisson population and the statistic $ns^2/\bar{x}$, which is sometimes called the index of dispersion. The conditional distribution of this statistic is studied, given that $\bar{x}>0$. The first four moments of this distribution are expressed in terms of the first three negative moments of a positive Poisson variate. A 10-place table for the moment of order -1 is given for integer values 1(1)50(5)125 of the parameter. This table overlaps in part with a recent table by E. L. Grab and I. R. Savage [J. Amer. Stat. Assoc. **49**, 169–177 (1954); MR **15**, 636]. *E. Lukacs.*

Bol'šev, L. N. A nomogram connecting the parameters of a normal distribution with the probabilities for classification into three groups. Inžen. Sb. **21**, 212–214 (1955). (Russian)

Let X be a normally distributed random variable with mean m and standard deviation σ and let c_1 and c_2 $(-\infty<c_1<c_2<+\infty)$ be two real numbers. The author constructs an alignment chart for the determination of the probabilities $P(X_1<c_1)$, $P(c_1<X<c_2)$ and $P(X>c_2)$. The nomogram is entered with σ/d and a/d where $a=m-(c_1+c_2)/2$ and $d=(c_2-c_1)/2$. *E. Lukacs.*

James, A. T. The non-central Wishart distribution. Proc. Roy. Soc. London. Ser. A. **229**, 364–366 (1955).

The distribution in the title is that of the sample variance-covariance matrix calculated from a sample of N independent observations on k-variate normal populations with a common variance-covariance matrix Σ but different means. The author observes that in order to evaluate this distribution it is enough to find that of $X'X$ given that the element of probability for X is

$$g(X'X)\exp(\operatorname{tr}\Sigma^{-1}M'X)\prod dx_{ij}, \tag{1}$$

where X is an $n\times k$ matrix variate, $n=N-1$, and the function g is known. He proves that the distribution of $X'X$ is unaltered if (1) is replaced by

$$g(X'X)\phi(X'X)\prod dx_{ij}, \tag{2}$$

where

$$\phi(X'X)=\int_{\mathfrak{H}}\exp(\operatorname{tr}H'Z)dV(H)=\Psi(Z), \tag{3}$$

say. Here $Z=X\Sigma^{-1}M'$ and H varies over the group $\mathfrak{H}$ of all orthogonal matrices, on which $V(H)$ is the invariant measure (with $V(\mathfrak{H})=1$ [cf. A. T. James, Ann. Math. Statist. **25**, 40–75 (1954); MR **15**, 726]. On the basis of (2) a known result [J. Wishart, Biometrika **20A**, 32–52 (1928)] then yields the distribution of $X'X$ in terms of g and ϕ. [See also the following review.]

H. P. Mulholland (Birmingham).

James, A. T. A generating function for averages over the orthogonal group. Proc. Roy. Soc. London. Ser. A. **229**, 367–375 (1955).

The author gives a method of evaluating $\Psi(Z)$ [for notation see the preceding review]. He shows that $\Psi(Z)=\xi(r_1,\cdots,r_n)$, where ξ is an everywhere-analytic function of the elementary symmetric functions $r_1,\cdots,r_n$ of the latent roots $\alpha_1,\cdots,\alpha_n$ of $X'X$ and satisfies the differential equations

$$4\sum_{\mu,\nu=1}^{n}a_{\mu\nu}^{(j)}\frac{\partial^2\xi}{\partial r_\mu\partial r_\nu}+2(n-j+1)\frac{\partial\xi}{\partial r_j}-\delta_{ij}\xi=0 \tag{*}$$

$(j=1,\cdots,n)$ with $a_{\mu\nu}^{(j)}=a_{\nu\mu}^{(j)}=\varepsilon r_{\mu+\nu-j}$ $(\mu\le\nu)$, where ε takes the values 1, 0, -1, 0 when j lies in the intervals $[1,\mu]$, $(\mu,\nu]$, $(\nu,\mu+\nu]$, $(\mu+\nu,n]$, respectively. The equations (*) can be solved by multiple power series in $r_1,\cdots,r_n$. For $\xi(r_1,r_2,r_3,0,\cdots,0)$ the coefficients in the triple power series are found explicitly.

The author notes that if $m(H)$ is any monomial in the elements of H the integral of $m(H)$ over $\mathfrak{H}$ can be found from $\Psi(Z)$ by equating coefficients of $m(Z)$ in the second of the equalities (3) in the preceding review, and he evaluates the integral explicitly when $m(H)$ is $h_{11}h_{12}h_{21}h_{22}h_{33}^2h_{44}^2$. *H. P. Mulholland* (Birmingham).

Paskevič, V. S. On a property of control charts used for running control. Prikl. Mat. Meh. **17**, 513–516 (1953). (Russian)

It is known that in a normal population any translation-invariant statistic is independent of the sample mean. D. Basu and R. G. Laha [Sankhyā **13**, 359–362; **14**, 180 (1954); MR **16**, 51] showed that the converse is in general not true. The question arises therefore what conditions a translation-invariant statistic, or a system of such statistics, must satisfy in order that its independence from the sample mean should imply the normality of the population. A partial answer, referring to systems of translation-invariant statistics was given by this reviewer [C. R. Acad. Sci. Paris **238**, 444–445 (1954); MR **15**, 542]. In the present paper the author deals with the problem for a single statistic. He gives the problem a geometric interpretation by introducing the concept of a tube statistic. A nonnegative statistic $T(X_1,X_2,\cdots,X_n)$ is said to be a tube statistic if the function $T(x_1,x_2,\cdots,x_n)$ satisfies the following conditions: (i) $T(x_1,x_2,\cdots,x_n)=0$ if and only if $x_1=x_2=\cdots=x_n$. (ii) The level surfaces $T(x_1,x_2,\cdots,x_n)=a$ are cylinders with the line $x_1=x_2=\cdots=x_n$ as their common axis. (iii) Any level surface $T(x_1,x_2,\cdots,x_n)=a$ can be obtained from the surface $T(x_1,x_2,\cdots,x_n)=1$ by a homothetic transformation with $x_1=x_2=\cdots=x_n$ as the axis and the homothetic ratio $f(a)$. The author proves the following theorem: Let $X_1,X_2,\cdots,X_n$ be a sample from a population with three-times differentiable distribution function. A tube statistic, satisfying certain smoothness conditions, too complicated to be listed here, is independent of the sample mean if, and only if, the population is normal. Every tube statistic is translation-invariant; however there are translation-invariant statistics, such as the third central sample moment, which are not tube statistics. Moreover, it is known (see the first reference) that the independence of the third central sample moment from the sample mean characterizes the normal population so that the interesting theorem proven in this paper gives only a partial answer to the problem.

E. Lukacs (Washington, D.C.).

Oderfeld, J. On sampling inspection with a two-sided criterion. Zastos. Mat. **2**, 210–224 (1955). (Polish. Russian and English summaries)

Let the variable X, used in acceptance inspection, have the normal distribution $N(\mu,\sigma)$, and let an item be considered defective if X is outside of a specified interval (D,G). The acceptance region considered is

$$D+ks\le\bar{X}\le G-ks,$$

where $\bar{X}$ and s are the sample mean and sample standard deviation; the possibility of a similar study of more general acceptance regions is mentioned. Let w denote the percentage defective and $P(w)$ the probability of accepting

a lot (the OC curve). For given n, k, D, G, this operating characteristic is obtained in a parametric form $P=g(c)$, $w=h(c)$, where $c=2\sigma/(G-D)$, and it is shown how practical problems may then be solved by numerical computation. *Z. W. Birnbaum* (Seattle, Wash.).

Weiss, Lionel. On confidence intervals of given length for the mean of a normal distribution with unknown variance. Ann. Math. Statist. **26**, 348–352 (1955).

A confidence interval of given length and with confidence coefficient at least β $(0<\beta<1)$, can be found for the mean of a normal distribution with unknown variance, using a two-stage sampling procedure. The center of the interval is best taken to be the mean of the combined sample. The size of the second sample is taken to be a function of the differences of observations in the first sample. Given any initial rule, another rule in which the size N of the second sample is an increasing function of the sample variance of the first sample can be constructed such that large values of N are less probable under the new rule. Stein's procedure [Ann. Math. Statist. **16**, 243–258 (1945); MR **7**, 213] makes N an increasing function of the first sample variance. His rule can be used as the initial rule needed in the method of the present paper. *S. W. Nash* (Vancouver, B.C.).

Sarhan, A. E. Estimation of the mean and standard deviation by order statistics. II. Ann. Math. Statist. **26**, 505–511 (1955).

[For part I see same Ann. **25**, 317–328 (1954); MR **15**, 809.] Best linear unbiased estimates based on order statistics are given for the mean and standard deviation of three distributions: a U-shaped one, a parabolic one, and a skewed one. Calculations are limited to the case where the sample size $n=2, 3, 4, 5$. Variances of the best linear estimates are given and the efficiency of various conventional estimates relative to the best linear estimates are computed. A general idea is given of how the weights assigned to the coefficients in the best linear estimate of the mean for symmetrical distributions change as the parent distribution changes. *B. Epstein* (Stanford, Calif.).

Ogawa, Junjiro, and Kogo, Kusunori. On the correlation of efficient estimates of unknown parameters. Osaka Math. J. **7**, 15–22 (1955).

Let α_1^* be an efficient estimator of a parameter α, and α_2^* a regular unbiased estimator of α of efficiency $e>0$. The significance of the terms is as in Cramér [Mathematical methods of statistics, Princeton, 1946, p. 482; MR **8**, 39], where a proof is given that the correlation coefficient between α_1^* and α_2^* is $\sqrt{e}$. The authors assert that this proof suffers from the defect that it makes use of the statement, which has never been proved, that a linear combination of regular estimates is regular. The authors give another proof, which does not use this statement.

Suppose α_1^* and α_2^* are two asymptotically normal estimators, of which the first is efficient in the sense of Fisher, i.e., has uniformly minimum asymptotic variance among a class of asymptotically normal estimators subject to certain regularity conditions. It was proved by Fisher and others that the correlation coefficient between α_1^* and α_2^* is $\sqrt{e}$, where e is the efficiency of α_2^*. It was pointed out to the reviewer by J. Wolfowitz that all these proofs involve the statement or implication that a linear combination of α_1^* and α_2^* is asymptotically normal, and that this statement is not in general true without assumptions on the joint asymptotic distribution of α_1^* and α_2^*. *M. Muller* (Ithaca, N.Y.).

Aggarwal, Om P. Some minimax invariant procedures for estimating a cumulative distribution function. Ann. Math. Statist. **26**, 450–463 (1955).

Suppose $X_1, X_2, \cdots, X_n$ are independently and identically distributed, with a common unknown continuous distribution function $F(x)$. Let $\hat{F}(x)$ be an estimate of $F(x)$ based on $X_1, X_2, \cdots, X_n$. The author considers estimates $\hat{F}(x)$ which are step-functions with jumps at $X_1, X_2, \cdots, X_n$ and nowhere else, and loss functions of the following types:

$$\int_{-\infty}^{\infty} |F(x)-\hat{F}(x)|^r dF(x),$$

$$\int_{-\infty}^{\infty} \frac{|F(x)-\hat{F}(x)|^r}{F(x)[1-F(x)]} dF(x).$$

Then the expected loss is independent of $F(x)$, and the author computes the minimax estimate of $F(x)$ among the class of estimates considered. *L. Weiss.*

Guttman, Louis. Reliability formulas for noncompleted or speeded tests. Psychometrika **20**, 113–124 (1955).

If a test containing m items be given to a set of subjects, it may happen, e.g. in the case of speed tests, that $m-n$ of the items are either answered correctly by all the subjects or are attempted by none. Then only the remaining n items contribute to the variance of test scores. In this and a recent paper [Psychometrika **18**, 225–239 (1953)] the author studies measures of the reliability of such tests to which the standard reliability formulas do not apply. For an infinite population of subjects and infinite repetitions of the test, using results from the earlier paper, two lower bounds for the reliability of such tests are derived for the case in which the score both for an incorrectly answered item and for an item not attempted is zero. It is also assumed that if a person attempts an item, his score on any later item is independent of that on the earlier one. The quantities appearing in these bounds may be estimated from a single trial of the test. Which of the two bounds is the better depends on the circumstances. *C. C. Craig.*

Tate, Robert F. The theory of correlation between two continuous variables when one is dichotomized. Biometrika **42**, 205–216 (1955).

Consider a real number ω and a drawing (X, Y) from a bivariate normal distribution of correlation ϱ. Replace the pair (X, Y) by a new pair (X, Z), in which Z is 0 or 1 according as $Y<\omega$ or not. Then for a sample of n pairs (X, Z) the author studies the problem of estimating ϱ. For this purpose he gives various properties of the maximum likelihood estimator, also of Pearson's biserial correlation coefficient. He finds the asymptotic efficiency of Pearson's coefficient to be 1 when $\varrho=0$ and 0 when $|\varrho|=1$, and concludes that Pearson's coefficient should be used only to test for low values of $|\varrho|$. It would seem that the case against Pearson's coefficient for large $|\varrho|$ is somewhat overstated. The transition is made from a proof that the asymptotic efficiency of this coefficient is zero to a statement that the coefficient is an asymptotically least efficient estimator and then to the conclusion that it is a "worst" estimator. There are three objections to this argument. First, the efficiency to which the proof applies is not an instance of the general concept of efficiency; it is a

relative efficiency. Now by the author's equation on page 208 it appears that the maximum likelihood estimate is particularly accurate at $|\rho|=1$, so that the comparative inefficiency of the Pearson coefficient does not preclude its being quite accurate here too. Second, even without this special accuracy of the maximum likelihood estimate at $|\rho|=1$, its variance falls off as $1/n$. The variance of another statistic might fall off almost as fast, e.g. as $(\log n)/n$. In this case the second statistic would have asymptotic efficiency zero, though it would clearly be far from the worst estimator one could devise — indeed, might be quite useful. Third, the "efficiency" refers to only one of our costs; it ignores the computing time. (The force of this last objection is mitigated in the present problem by the fact that the calculation of the maximum-likelihood statistic can easily be programmed for high-speed computing machinery, as the author indicates.) *A. Blake.*

Rao, C. Radhakrishna. Estimation and tests of significance in factor analysis. Psychometrika **20**, 93–111 (1955).

The author defines factor analysis as a group of methods based on the model $x_i=z_i+s_i$ $(i=1, \cdots, p)$, in which the x_i are (unreplicated) measurements of p properties of an individual, the z_i are hypothetical (causative) factors common to the individuals, and the s_i are hypothetical specific factors, i.e. factors not common to the individuals. This model differs from those in Hotelling's component analysis. The author considers the estimation of the rank of the space of the vectors z, and the testing of hypotheses concerning this rank. He turns next to the problem of finding an optimal sequence of vectors determining such a space. For the determination of each vector to be adjoined in forming the sequence he considers two criteria. First, a least-squares condition, leading to the equations of principal factor analysis. Second, a condition of maximum correlation between z and x, leading to a theory which he designates as canonical factor analysis. In the normal case the latter theory yields maximum-likelihood estimates. The author works out the equations for the two methods. The equations for the canonical factor analysis are being programmed for the Illiac digital computer. *A. Blake.*

★ **Wald, Abraham. Testing the difference between the means of two normal populations with unknown standard deviations.** Selected papers in statistics and probability by Abraham Wald, pp. 669–695. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1955. $8.00.

Let x_{ij}, $i=1, 2$; $j=1, \cdots, N$, be independent normally distributed chance variables with mean μ_i and variance σ_i^2, unknown to the statistician. The problem is to test the hypothesis H: $\mu_1=\mu_2$. The author imposes certain conditions of sufficiency and invariance on the test, whose effect is that the critical region must be defined by an inequality of the form

$$\left| \frac{y}{\sqrt{(s_1^2+s_2^2)}} \right| \geq \varphi(l),$$

where $\bar{x}_i=N^{-1}\sum_j x_{ij}$, $s_i^2=(N-1)^{-1}\sum_j (x_{ij}-\bar{x}_i)^2$, $y=N^{\frac{1}{2}}(\bar{x}_1-\bar{x}_2)$, $l=s_2^2/s_1^2$, and φ is a function to be assigned. The size α of the critical region is a function of $\lambda=\sigma_2^2/\sigma_1^2$ and, of course, of φ; the author requires that, for all λ, $\alpha(\lambda|\varphi)\leq\alpha_0$ for given α_0. When φ is identically a constant, the resulting test is asymptotically most powerful unbiased, but is not satisfactory for small N.

The author therefore studies

$$\varphi(l)=\varphi_0(l)=c-c^1 l(1+l)^{-2}$$

with c and c^1 constants, whose size is almost constant (with λ) even for small N. It is shown that, for all λ,

$$\alpha(\lambda|\varphi_0)-\alpha(0|\varphi_0)=O(N^{-2}).$$

The critical region corresponding to φ_0 is uniformly more powerful than that corresponding to $\varphi=$constant for the same maximum size of test.

Let A be a chance variable with the usual F distribution with (n, n) degrees of freedom. One has that

$$\alpha(\lambda|\varphi)=EG(\varphi(\lambda A), \lambda, A),$$

where G is a function defined in the paper. Let $B=\min(A, c_0)$ for some positive constant c_0. Define

$$\alpha^*(\lambda|\varphi)=EG(\varphi(\lambda B), \lambda, B).$$

The author proves that, for any $c_0>0$, and for any positive $\beta<1$, there exists exactly one analytic function $\varphi_\beta(l)$ which is regular at $l=0$ and such that $\alpha^*(\lambda|\varphi_\beta)=\beta$ for all λ not greater than a certain positive λ_0. A method is given of computing the coefficients of the series for φ_β in powers of l.

The problem naturally arises, and is mentioned by the author, whether there exists a function $\varphi(l)$ such that $\alpha(\lambda|\varphi)$ is a constant identically in λ, or a function $h(s_1^2, s_2^2)$ such that $\Pr\{y>h(s_1^2, s_2^2)\}$ is a constant identically in (σ_1^2, σ_2^2) when $\mu_1=\mu_2$. A method of computing the coefficients of the power series expansion of h was given by Welch [Biometrika **34**, 28–35 (1947); MR **8**, 394]. The latter's argument is vitiated by a serious lacuna, namely that the existence of a function h has never been proved. The same flaw affects the following papers which develop Welch's orginal argument further: Welch, Ann. Math. Statist. **18**, 118–122 (1947); Biometrika **38**, 330–336 (1951); MR **8**, 474; **13**, 762; James, ibid. **38**, 324–329 (1951); MR **13**, 762. *J. Wolfowitz* (Ithaca, N.Y.).

Kac, M., Kiefer, J., and Wolfowitz, J. On tests of normality and other tests of goodness of fit based on distance methods. Ann. Math. Statist. **26**, 189–211 (1955).

Let x_i $(i=1, \cdots, n)$ be independent random variables with the same cumulative distribution function (c.d.f.) $G(x)$, and $G_n^*(x)$ be the empirical c.d.f. of $x_1, \cdots, x_n$. The authors study the use of various kinds of distance between $G_n^*(x)$ and a given class of c.d.f.'s to test whether $G(x)$ belongs to this class. For any two c.d.f.'s $F(x)$ and $H(x)$ let $\delta(F, H)=\sup_x |F(x)-H(x)|$. Let $N(x)$ be the normal c.d.f. agreeing with $G_n^*(x)$ in mean and variance. In order to test normality the authors investigate especially the sampling distributions of the statistics $v_n=\delta[G_n^*(x), N(x)]$ and $w_n=\int[G_n^*(x)-N(x)]^2 dN(x)$. They show that as $n\to\infty$ the c.d.f. of nw_n tends to that of $W=\int_0^1[Z(t)]^2dt$, where $Z(t)$, $0\leq t\leq 1$, denotes the sample function (which is continuous with probability 1) of a Gaussian process whose covariance function (c.f.) is of the form

$$\min(s, t)-st-h_1(s)h_1(t)-h_2(s)h_2(t):$$

explicit formulae are given for h_1 and h_2. The process with c.f. $\min(s, t)-st$ was treated by M. Kac [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability 1950, Univ. of California Press, 1951, pp. 189–215; MR **13**, 568]. The authors therefore develop and apply (twice) a general method of constructing a new process from a given one so as to diminish the c.f. by a term $h(s)h(t)$. They find that the characteristic function of W is of the form $\prod_1^\infty (1-2it\lambda_j)^{-\frac{1}{2}}$ and compute $\lambda_1, \cdots, \lambda_8$, whence they obtain an approximation to the c.d.f. of W. This is compared with the results of sampling experiments: for $n=100$ good

agreement is found in the upper tail. Experimental estimates are given also for the c.d.f. of $n^{\frac{1}{2}}v_n$, and a table of (estimated) common percentage points is given for both nw_n and $n^{\frac{1}{2}}v_n$, with $n=25$ or 100.

Various other tests are discussed. In particular, when $F(x-\theta)$ is the c.d.f. of the rectangular distribution on $(\theta-\frac{1}{2}, \theta+\frac{1}{2})$, and R is the class of such c.d.f.'s, they define $\delta(G_n^*, R)$ to be $\inf_\theta \delta(G_n^*, F(x-\theta))$, and deduce from results given by J. L. Doob [Ann. Math. Statist. **20**, 393–403 (1949); MR **11**, 43] that the limiting c.d.f. of $u=n^{\frac{1}{2}}\delta(G_n^*, R)$ equals $1-\sum_1^\infty (32m^2u^2-2)\exp(-8m^2u^2)$.

Lastly, for tests of a simple hypothesis concerning $G(x)$, they show that the asymptotic power of the Kolmogorov test, and of the classical ω^2 test, is considerably greater than that of the optimum χ^2 test, and that the power of the tests (of composite hypotheses) based on v_n or w_n is similar to that of the Kolmogorov and ω^2 tests.

H. P. Mulholland (Birmingham).

Huron, Roger. Loi multinomiale et test du χ^2. C. R. Acad. Sci. Paris **240**, 2047–2048 (1955).

The author gives a linear transformation which replaces multinomial variates by uncorrelated variates with means 0, variances 1. These can be used to analyze the chi square of the goodness-of-fit test into component chi squares, though the author does not say this too plainly.

S. W. Nash (Vancouver, B.C.).

Hodges, J. L., Jr. A bivariate sign test. Ann. Math. Statist. **26**, 523–527 (1955).

Observations (x, y) are made on n pairs of items, one item of each pair drawn from each of two populations. The alternative considered to the hypothesis that both populations are identically distributed is that the center of the second population is displaced from the center of the first in an unspecified direction. Let S be the number of differences $(x'-x, y'-y)$ whose projections onto a given vector are positive. The statistic used is M, the maximum of S over all possible directions for the given vector. The distribution of M under the null hypothesis is derived for the case $M>\frac{3}{4}n$. Five-place tables of $P(M\geq m)$ are given for $n=1(1)30$ and $m>\frac{3}{4}m$.

S. W. Nash.

van Eeden, Constance, and Hemelrijk, J. A test for the equality of probabilities against a class of specified alternative hypotheses, including trend. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17**, 191–198, 301–308 (1955).

The authors consider a sequence of experiments, where the νth experiment consists of k_ν independent (binomial) chance variables $\alpha_{i,\nu}$, $i=1, \cdots, k_\nu$, with

$$P\{\alpha_{i,\nu}=j\}=\binom{n_{i,\nu}}{j}p_{i,\nu}^{\,j}(1-p_{i,\nu})^{n_{i,\nu}-j}$$

for $0\leq j\leq n_{i,\nu}$. The n's are given constants, the p's are unknown to the statistician. The problem is to test H_∞, the hypothesis that $p_{1,\nu}=\cdots=p_{k_\nu,\nu}$ by a test which will be consistent only when $\lim\inf_{\nu\to\infty}|\theta_\nu|>0$, where $\theta_\nu=\sum_i g_{i,\nu}p_{i,\nu}$ and the g's are known constants subject to the normalizing conditions $\sum_i g_{i,\nu}=0$, $\sum_i |g_{i,\nu}|=1$.

Let $n_\nu=\sum_i n_{i,\nu}$, $t_{1,\nu}=\sum_i \alpha_{i,\nu}$, $t_{2,\nu}=n_\nu-t_{1,\nu}$. The authors' test statistic is

$$\frac{1}{s}\sum_i n_{i,\nu}^{-1}g_{i,\nu}\alpha_{i,\nu},$$

where s is the positive square root of

$$[n_\nu(n_\nu-1)]^{-1}t_{1,\nu}t_{2,\nu}\sum_i n_{i,\nu}^{-1}g_{i,\nu}^2.$$

One-sided and two-sided tests are used. The authors prove that a certain conditional asymptotic distribution of the test statistic under H_0 and under certain conditions on the constants is normal and that, under the latter conditions, the test is consistent essentially only when $\lim\inf\theta_\nu>0$.

J. Wolfowitz (Ithaca, N.Y.).

Putter, Joseph. The treatment of ties in some nonparametric tests. Ann. Math. Statist. **26**, 368–386 (1955).

The occurrence of ties in nonparametric tests is usually handled in one of two ways: by introducing an additional chance process to break the ties or by using midranks (or a similar procedure). The author is concerned with the effect of this "randomized" and "nonrandomized" way of treating ties on the performance characteristic of the one-sided sign test and the Wilcoxon test. It is shown that randomization reduces the exact power of the sign test and the asymptotic efficiency of the sign test and the Wilcoxon test.

G. E. Noether (Boston, Mass.).

Ehrenfeld, Sylvain. On the efficiency of experimental designs. Ann. Math. Statist. **26**, 247–255 (1955).

Let $\mathbf{y}$ be a vector of independent and normally distributed random variables with $E(\mathbf{y})=\mathbf{X}\boldsymbol{\beta}$ and $E(\mathbf{yy}')=\sigma^2\mathbf{I}$, where $\boldsymbol{\beta}$ is unknown and the elements x_{ai} of the "design matrix" $\mathbf{X}$ are chosen in a given domain T. Let $\hat{\theta}$ be the best unbiased linear estimate, based on $\mathbf{y}$, of $\theta=\mathbf{t}'\boldsymbol{\beta}$, where $\mathbf{t}$ is $\neq\mathbf{0}$ and such that the equation $\mathbf{X}'\mathbf{X}\boldsymbol{\rho}=\mathbf{t}$ has solutions for $\boldsymbol{\rho}$. Let L denote the length of the usual confidence interval for θ, and let H_0 be the hypothesis that $\boldsymbol{\beta}=\mathbf{0}$. The author proves the inequality

$$(*) \qquad \lambda_{\max}^{-1}\leq\boldsymbol{\rho}'\mathbf{X}'\mathbf{X}\boldsymbol{\rho}/\mathbf{t}'\mathbf{t}\leq\lambda_{\min}^{-1},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and the minimum nonzero characteristic roots of $\mathbf{X}'\mathbf{X}$. Using (*) he gives, for the case when $(\mathbf{X}'\mathbf{X})^{-1}$ exists, bounds on Var $(\hat{\theta})$ and $E(L)$, and also on the parameter $\phi=\boldsymbol{\beta}'(\mathbf{X}'\mathbf{X})^{-1}\boldsymbol{\beta}/\sigma^2$, on which the power function of the usual F test of H_0 depends monotonically. After pointing out that these bounds are attainable, and that the upper bounds of Var $(\hat{\theta})$ and $E(L)$ decrease and the lower bound of ϕ increases when $\lambda_{\min}$ increases the author defines the efficiency of a design as the ratio $\lambda_{\min}/u$, where u is the maximum of $\lambda_{\min}$ for x_{ai} in T. He calls a design most efficient when $\lambda_{\min}/u=1$. For two general cases of T (where $\mathbf{X}$ need not be of full rank) he shows how most efficient designs can be determined. In the first case the diagonal elements of $\mathbf{X}'\mathbf{X}$ are fixed. In the second case some of the column vectors of $\mathbf{X}$ are fixed. The author applies his results to some design problems and shows, e.g., that the Latin square is a most efficient design.

D. M. Sandelius (Göteborg).

Bradley, Ralph Allan, and Terry, Milton E. Rank analysis of incomplete block designs. I. The method of paired comparisons. Biometrika **39**, 324–345 (1952).

The following experiment is considered. There are t treatments which are compared subjectively in n repetitions of all $t(t-1)/2$ possible pairs (blocks). In each pair a rank of 1 is given to the preferred treatment and a rank of 2 to the other treatment.

It is assumed that the treatments have true ratings (or preferences) $\pi_1, \cdots, \pi_t$ on a particular subjective continuum throughout an experiment. It is required that every $\pi_i\geq 0$ and, for convenience, that $\sum\pi_i=1$. Further, when treatments i and j are compared, it is assumed that

the probability that treatment i will be preferred is $\pi_i/(\pi_i+\pi_j)$. A likelihood ratio test statistic B is developed for testing the null hypothesis H_0 that $\pi_1=\cdots=\pi_t$ against the alternative hypothesis H_1 that the π's are identical within each of m groups ($m\leq t$), but may differ from group to group. Equations are developed from which the maximum-likelihood estimates of the π's (denoted by p's) can be calculated.

Two special cases are considered: (i) $m=t$ and (ii) $m=2$. The corresponding test statistics are denoted by B_1 and B_2. Because in case (i) the maximum-likelihood equations for the p's are difficult to solve, the authors have prepared a table (Appendix A) for estimating them. The table is entered with the t sums of ranks for the treatments. The p's are read from the table, together with the corresponding value of B_1 and the significance level P, which is the probability of obtaining a value of B_1 as small as, or smaller than, the observed value (under H_0 small values of B_1 are associated with discordant sets). The table is for $t=3$, $n=1, \cdots, 10$ and $t=4$, $n=1, \cdots, 6$. It is mentioned that the tables for B_1 may be used to determine the distribution of B_2, and that tables for B_2 will be prepared at a later date.

Consideration is given to the combination of experiments. If there are g groups of experiments with repetitions n_u ($u=1, \cdots, g$), $\sum n_u=n$, then it sometimes is desirable to postulate parameters $\pi_{1u}, \cdots, \pi_{tu}$, $\sum_i \pi_{iu}=1$ for the uth group, admitting the possibility that the π's differ from group to group. The appropriate "combined" analysis utilizes a statistic $B_1^c=\sum B_1^u$, where B_1^u is B_1 for the uth group. Values of B_1^c significant at the .10 level of significance or higher are tabled (Appendix B) for 2, 3, 4 and 5 equal groups of 2, 3, 4 and 5 repetitions each. A coefficient of agreement among groups is given as $B_1-B_1^c$, where B_1 is computed by pooling all n experiments as if there are no groups. Because for small samples the distribution of $B_1-B_1^c$ under H_0 is not parameter-free, a conditional parameter-free test is discussed. Large sample statistics, based on the general theory for likelihood ratio tests, are presented. Finally, an illustrative example is given. *W. S. Connor* (New Brunswick, N.J.).

Bradley, Ralph Allan. Rank analysis of incomplete block designs. II. Additional tables for the method of paired comparisons. Biometrika **41**, 502–537 (1954).

The paper contains additional tables for the rank analysis of paired comparisons. The tables are the extension of Appendix A of the paper by R. A. Bradley and M. E. Terry reviewed above to 4 treatments with 7 and 8 repetitions and 5 treatments with 1, $\cdots$, 5 repetitions of the balanced incomplete block design with $k=2$ and $\lambda=1$. There is a summary of four uses of a statistic B_1, which is included in the tables, and a discussion of the accuracy of the tables, computing checks, and two errors in the previously published tables, cited above. *W. S. Connor.*

Johnston, J. A revised test for systematic oscillation. J. Roy. Statist. Soc. Ser. B. **16**, 292–295 (1954).

The author points out that if the trend in a time series is estimated by application of a moving average operator T, then the operator with which deviations from trend are estimated is $1-T$, and that this fact must be borne in mind when using Kendall's test for systematic oscillation. *P. Whittle* (Wellington).

See also: Bahadur, p. 22.

Theory of Games, Mathematical Economics

Birch, B. J. On games with almost complete information. Proc. Cambridge Philos. Soc. **51**, 275–287 (1955).

A game with perfect information has an equilibrium-point of pure strategies; this was first proved for two-person games by Zermelo [Proc. Fifth Internat. Congress Math., Cambridge, England, 1912, vol. 2, Cambridge, 1913, p. 501] and extended to n-person games by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **36**, 570–576 (1950); MR **12**, 515]. More recently, Dalkey [Contributions to the theory of games, v. 2, Princeton, 1953, pp. 217–243; MR **14**, 891] and Otter and Dunne [Proc. Nat. Acad. Sci. U.S.A. **39**, 310–314 (1953); MR **14**, 891] have proved the stronger result: If in the complete inflation of a game Γ every player has complete information about every other player, then Γ has an equilibrium-point of pure strategies.

The present paper extends this result by defining a game Γ to have almost complete information for a given player if he has complete information about every player, and every other player has complete information about him. Theorem. If Γ is any game and $\bar{\Gamma}$ is its complete inflation, there is an equilibrium-point of Γ at which those players for whom $\bar{\Gamma}$ has almost complete information have pure strategies. (A slightly restricted converse is also shown.) The proof is based on a theory of the decomposition of games initiated by the reviewer [Contributions to the theory of games, v. 2, Princeton, 1953, pp. 193–216; MR **14**, 999]. A key result obtained in this paper asserts that every game structure can be completely decomposed into indecomposable components; this decomposition is unique and leads to a tree of substructures. Another result, too complicated to reproduce here, characterizes the equilibrium-points of a game structure in terms of those of a subgame and a difference game. A sketch of a decomposition theory relevant to behavior strategies is also presented. *H. W. Kuhn* (Bryn Mawr, Pa.).

Stein, Charles. A necessary and sufficient condition for admissibility. Ann. Math. Statist. **26**, 518–522 (1955).

Let $K(a, b)$ be the pay-off matrix of a zero-sum two-person game. The main theorems are the following: 1) A pure strategy b_0 of the second (minimizing) player is admissible if, for any pure strategy a of the first player and any $\varepsilon>0$, there exists a (possibly mixed) strategy ξ of the first player and δ, $0<\delta\leq 1$, such that b_0 is $\varepsilon\delta$-Bayes with respect to the strategy of the first player which mixes ξ and a with probabilities $(1-\delta)$ and δ, respectively. 2) Under certain additional assumptions this condition is necessary for b_0 to be admissible. *J. Wolfowitz.*

Isbell, J. R. A class of game solutions. Proc. Amer. Math. Soc. **6**, 346–348 (1955).

This note presents a class of finite von Neumann-Morgenstern solutions to certain n-person games. They are pseudo-main solutions (a concept introduced by H. M. Gurk), i.e., are defined by a family F of sets of players and constants x_i ($i=1, \cdots, n$) by the rule: there is a one-to-one correspondence between sets $S \in F$ and imputations $\alpha(S)$ of the solution such that $\alpha_i(S)=x_i$ for $i \in S$ and $\alpha_i(S)=0$ otherwise. The games Γ involved are such that every $(n-k+1)$-person set contains a k-person winning set for some k. If A is the set of all k-person winning sets and B is the set of all k-person sets not in A whose complements do not contain elements of A, then $F=A\cup B$ and $x_i=1/k$ define a pseudo-main solution for Γ.

H. W. Kuhn (Bryn Mawr, Pa.).

Dantzig, George B. Upper bounds, secondary constraints, and block triangularity in linear programming. Econometrica **23**, 174–183 (1955).

In this paper, short-cut computational methods are developed for solving linear programs in which the technology matrix can be partitioned into submatrices so that the nonzero elements appear in the following block-triangular form:

$$\begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \cdot & \cdot & \cdots & \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}.$$

The algorithms are modifications of the simplex method and apply to situations in which the non-negative variables also have upper bounds or are subject to "secondary" constraints of the form

$$\sum_j a_{kj}x_j + x_{n+k} = b_k \quad (k=1, \cdots, m';\ x_{n+k} \geq 0;\ j=1, \cdots, n).$$

H. W. Kuhn (Bryn Mawr, Pa.).

Dantzig, G., Fulkerson, R., and Johnson, S. Solution of a large-scale traveling-salesman problem. J. Operations Res. Soc. Amer. **2**, 393–410 (1954).

The Traveling Salesman Problem can be described as follows: Given an n by n matrix $D=(d_{ij})$, where d_{ij} represents the "distance" from i to j, arrange the integers $1, \cdots, n$ in a cyclic order so that the sum of the d_{ij} between consecutive integers is minimal. (Of course, for an actual salesman, the matrix is symmetric.) Various unsuccessful attempts have been made to pose an equivalent linear program. At present, the programs are known, in general, only for $n \leq 5$, and for $n \leq 7$ in the symmetric case, and the number of constraints increases very rapidly with n (unpublished work of I. Heller, R. Z. Norman, and the reviewer). The present paper shows that a certain tour of 49 cities, one in each of the 48 states and Washington, D.C. has the shortest road distance. The algorithm used is the simplex method applied to a fortuitous selection of constraints; since the answer obtained satisfies the constraints which are not used (or known), it is the optimal tour. *H. W. Kuhn* (Bryn Mawr, Pa.).

Wong, Y. K. An elementary treatment of an input-output system. Naval Res. Logist. Quart. **1** (1954), 321–326 (1955).

An input-output system in which the outputs dominate the inputs can be described by the system of linear equations: $x_i - \sum_j a_{ij}x_j = h_i$ (i and $j=1, \cdots, n$) where $a_{ij} \geq 0$, $a_{ii} < 1$, $h_i > 0$, and $\sum_i a_{ij} \leq 1$ for all i and j. Theorem: There exist unique $x_i > 0$ satisfying this system if and only if $\det(I-A) \neq 0$. The methods employed are elementary (e.g., Cramer's rule and Cauchy's expansion of a determinant) and center about an analysis of the adjoint matrix of the matrix of coefficients. *H. W. Kuhn.*

Farquharson, Robin. Sur une généralisation de la notion d'équilibrium. C. R. Acad. Sci. Paris **240**, 46–48 (1955).

The author defines an equilibrium-point of order r in an n-person game to be a point in strategy space at which no set of r players can improve their individual positions by any changes in their strategies. Theorem I. An equilibrium-point of order 1 is a Nash equilibrium-point [Ann. of Math. (2) **54**, 286–295 (1951); MR **13**, 261]. In a "jeu d'ophélimité" it is assumed that, whenever player k prefers stategy point u to v, some player j prefers v to u. An outcome is a maximal set of strategy points which are indifferent to all players. Theorem II. In a two-person jeu d'ophélimité, all equilibrium-points of order 1 belong to the same outcome. Theorem III. In every outcome of a simple majority game with n players, there exists a strategy point u such that, for all $r < \frac{1}{2}n$, u is an equilibrium-point of order r. The results are stated in terms of an abstract preference relation, which is defined for all pairs of strategies and is transitive for each player.

H. W. Kuhn (Bryn Mawr, Pa.).

Bellman, Richard. Bottleneck problems, functional equations, and dynamic programming. Econometrica **23**, 73–87 (1955).

This paper, which is essentially expository in nature, describes the solution of "bottleneck" problems by the methods of dynamic programming. These have been considered in a previous note by the author [Proc. Nat. Acad. Sci. U.S.A. **39**, 947–951 (1953); MR **15**, 888]. Of particular interest in the present version is the use of a dual program both to verify and to derive a solution. *H. W. Kuhn.*

Charnes, A., Cooper, W. W., and Mellon, B. A model for optimizing production by reference to cost surrogates. Econometrica **23**, 307–323 (1955).

The mathematical content of this paper is mostly contained in the following interesting theorem: If

(1) $f(x)$ is continuous, convex and nondecreasing,

(2) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$,

(3) $a_m/m \geq a_j/j$,

for all $j=1, \cdots, n$, and m is the largest integer which satisfies this relation; then there is an optimal solution to minimizing $C(x_1, \cdots, x_n) = \sum_{i=1}^{n} f(x_i)$ subject to

$$\sum_{i=1}^{r} x_i \geq a_r \quad (r=1, \cdots, n),$$

$x_i \geq 0$ ($i=1, \cdots, n$) with the properties $\sum_{i=1}^{m} x_1 = a_m$, and $x_i = a_m/m$ ($i=1, \cdots, m$). Though this theorem seems only to define the solution for the first m variables x_i, it actually defines it completely. For $x_{m+1}, \cdots, x_n$ are then to be chosen so as to minimize $\sum_{i=m+1}^{n} f(x_i)$ subject to $\sum_{i=m+1}^{r} x_i \geq a_r - a_m$ ($r=m+1, \cdots, n$). Hence, the solution to the minimization problem is a series of stages in each of which x_i is constant for some set of consecutive values of i. The remarkable fact is that the solution is the same for varying functions $f(x)$ provided only that they satisfy (1).

The theorem has the following application to the behavior of a firm: Let x_i be the output in time period i, a_i the cumulated prediction requirements of the first i time periods, and $f(x)$ the cost of producing x units in a single time period; then the above theorem defines the minimum-cost solution (in the absence of inventory costs). The solution is independent of the exact nature of the cost function, and hence it is unnecessary to estimate the costs in order to reach an optimal production schedule, thereby decreasing considerably the informational demands upon the firm. [The authors assert this method can be applied to production of several commodities, but the reviewer believes there is an error in their procedure.]

K. J. Arrow (Stanford, Calif.).

Vazsonyi, Andrew. The use of mathematics in production and inventory control. Management Sci. **1**, 70–85 (1954).

Vazsonyi, Andrew. The use of mathematics in production and inventory control. II. Theory of scheduling. Management Sci. **1**, 207–223 (1955).

See also: Choquet, p. 62.

Mathematical Biology

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. III_1, III_2, III_3, III_4. Further discussions on cross-breeding. Proc. Japan Acad. **27**, 459–465, 466–471, 472–477, 478–483 (1951).

[For parts I–II see same Proc. **27**, 371–377, 378–383, 384–387 (1951); MR **13**, 964.] This paper considers the Mendelian inheritance of a system of m multiple alleles A_i $(i=1, 2, \cdots, m)$ at a single diploid locus in an infinite population X which results from partial cross-breeding between two initially separate populations X' and X'' of relative sizes λ' and λ'' (where $\lambda'+\lambda''=1$). Formulae for the frequencies of the genotypes A_iA_i and A_iA_j in the nth generation are derived in terms of λ' and λ'', the initial gene frequencies in X' and X'', and certain constants which define the degree of cross-breeding between X' and X''. Some results concerning the frequencies of phenotypes are derived as an application of these formulae. The formulae are also extended to cover the case of continuously overlapping generations. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. IV_1, IV_2, IV_3, IV_4, IV_5, IV_6, IV_7, IV_8. Mother-child combinations. Proc. Japan Acad. **27**, 587–592, 593–597; 598–603, 605–610, 611–614, 615–620 (1951); **29**, 68–71, 72–77 (1953).

The Mendelian segregation of a system of m alleles A_i $(i=1, 2, \cdots, m)$ is considered in the case of random mating of a mother from population X with a father from population X', these populations being each separately in equilibrium under internal random mating, p_i and p_i' being the respective frequencies of the gene A_i. Expressions are obtained in terms of the p's and p''s for $\pi(ij, hk)$ the probability of the combination A_iA_j mother with A_hA_k child.

For a single population in equilibrium under random mating expressions are given for $\pi(ij; hk, fg)$, the probability of the combination A_iA_j mother, A_hA_k and A_fA_g children, for the case when the children have the same father and for the case when the fathers are different. Bayes' theorem is applied to give the posterior probability of the mother being A_iA_j when the genotypes of the offspring are known. The formula for $\pi(ij; hk, fg)$ is generalised to apply to the set of genotypes represented by a mother and her n children by a common father.

Results for mother-children combinations are illustrated by detailed tables for the ABO and MN blood group systems. *A. R. G. Owen* (Cambridge, England.).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. V_1, V_2, V_3. Brethren combinations. Proc. Japan Acad. **27**, 689–693, 694–699 (1951); **29**, 78–82 (1953).

Expressions are derived for $\sigma(hk, fg)$, the probability that a pair of full sibs shall have genotypes A_hA_k, A_fA_g where A_i $(i=1, 2, \cdots, m)$ is a system of Mendelian alleles segregating in an infinite population in equilibrium under random mating. The result is extended to the case of any number n of sibs. Pairs of half sibs are also treated.

A. R. G. Owen (Cambridge, England).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. VI. Rate of danger in random blood transfusion. Proc. Japan Acad. **28**, 54–58 (1952).

This paper evaluates the probability of incompatibility of the blood types of two individuals chosen at random from an infinite population in equilibrium under random mating (a) for the ABO system (b) for the Qq± system. The worst distributions of phenotypes maximising the risk of a harmful blood reaction, are shown to be

(a) O 14.94%, A 33.12%, B 33.12%, AB 18.82%

(b) Q 50%, q+ 50%, q− absent.

A. R. G. Owen (Cambridge, England).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. VII_1, VII_2, VII_3, VII_4, VII_5, VII_6. Non-paternity problems. Proc. Japan Acad. **28**, 102–104, 105–108, 109–111, 112–115, 116–120, 121–125 (1952).

This paper evaluates $V(ij; hk)$, the probability that a man taken at random from an infinite population in equilibrium under random mating shall be of such a genotype as to prove that he cannot be the father of an A_hA_k child by an A_iA_j mother, where A_i $(i=1, 2, \cdots, m)$ is a system of allelomorphic genes. Various marginal distributions are derived; for example, the probability that a man taken at random can be proved not to be the father of any child of an A_iA_j mother. The results are extended to the case where the mother and the putative father belong to different populations each of which is separately in equilibrium under internal random mating. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. $VIII_1$, $VIII_2$, $VIII_3$. Further discussions on non-paternity problems. Proc. Japan Acad. **28**, 162–164, 165–168, 169–171 (1952).

This paper considers married couples comprised of an A_iA_j husband and an A_hA_k wife belonging to an infinite population in equilibrium under random mating in which a system of allelomorphic genes A_i $(i=1, 2, \cdots, m)$ is segregating. It derives the probability $U(ij, hk)$ that the husband will be able to assert his non-paternity in respect of a child produced by the wife and fathered by a man chosen at random from the population. Various other probabilities of interest are calculated. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. IX_1, IX_2, IX_3, IX_4. Non-paternity concerning mother-children combinations. Proc. Japan Acad. **28**, 207–212, 213–217, 218–223, 224–229 (1952).

This paper considers a mother of genotype A_iA_j having children A_hA_k and A_fA_g by the same father in an infinite population in equilibrium under random mating, A_i being an allelomorphic series. Expressions are derived for the probability that a man chosen at random will be able to assert his non-paternity in respect of (a) one of the children, (b) both of the children. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. X_1, X_2, X_3. Non-paternity concerning mother-child-combinations. Proc. Japan Acad. **28**, 249–253, 254–258, 259–264 (1952).

Given a mother of genotype A_iA_j and children A_hA_k and A_fA_g by possibly different fathers, expressions are derived for the probability that a man chosen at random will be able to assert his non-paternity in respect of (a) one of the children (b) both of the children. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. XI_1, XI_2. Absolute non-paternity. Proc. Japan Acad. **28**, 311–316, 317–322 (1952).

Given two children of the same mother having genotypes A_iA_j, A_kA_l, expressions are derived for the probability that a man chosen at random can (without reference to the mother's genotype) assert his non-paternity in respect of (a) one of the children (b) both of the children. An expression is also given for the probability that the father of the first child can assert his non-paternity in respect of the second child. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. XII_1, XII_2. Probability of paternity. Proc. Japan Acad. **28**, 359–364, 365–369 (1952).

A series A_i ($i=1, 2, \cdots, m$) of allelomorphic genes segregating in an infinite population in equilibrium under random mating is considered. Bayes' theorem is applied to obtain the posterior probability that a man of genotype A_aA_b is the father of an A_kA_l child when the mother's genotype is (a) known (b) unknown. Results are also given for the case where the genotypes of two children and their mother are known. *A. R. G. Owen.*

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. $XIII_1$, $XIII_2$. Estimation of genotypes. Proc. Japan Acad. **28**, 432–437, 438–443 (1952).

Bayes' theorem is applied to calculate the posterior probability of an individual being of an assigned genotype in respect of the blood-group systems Qq± and ABO, the phenotypes of n offspring of the individual being known. The calculation is performed on the assumption of random mating in an infinite population. Two cases are considered (a) phenotype of spouse, known (b) phenotype of spouse, unknown. *A. R. G. Owen* (Cambridge, England).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. XIV. Decision of biovular twins. Proc. Japan Acad. **28**, 444–449 (1952).

In a previous paper [parts V_1, V_2 of this series, reviewed ninth above] the author derived an expression for the probability that a pair of sibs should have different genotypes in respect of a system of Mendelian allelomorphs A_i ($i=1, 2, \cdots, m$) segregating in an infinite population in equilibrium. This expression is now applied with reference to the MN and ABO blood-group systems to obtain the probability that a pair of twins can be proved to be definitely biovular and not monovular.

A. R. G. Owen (Cambridge, England).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. XV. Detection of interchange of infants. Proc. Japan Acad. **28**, 517–520, 521–526, 527–532, 533–537 (1952).

This paper considers an allelomorphic series A_i ($i=1, 2, \cdots, m$) in an infinite population in equilibrium under random mating and calculates the probability that given two pairs of parents each with a child it will be possible to assert in the case of at least one of the children that it cannot be the child of that pair which are not its own parents. By summation an expression is derived for the probability of an interchange of infants (such as occurs occasionally in maternity hospitals) being detectable by determining the genotypes of the infants and their putative parents.

Thus the paper generalises results obtained by A. S. Wiener [J. Immunol. **24**, 443–454 (1933); Z. Indukt. Abstam. Vererbungslehre **59**, 229–235 (1931)].

A. R. G. Owen (Cambridge, England).

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Ganea, Tudor. Zur Multikohärenz topologischer Gruppen. II. Math. Nachr. **13**, 9–18 (1955).

[For part I see Math. Nachr. **7**, 323–334 (1952); **9**, 384 (1953); MR **14**, 136.] For a connected, locally connected topological group G, the inequality $r(G)\leq 1$ is shown to hold if G is either locally pathwise connected or locally unicoherent. *R. H. Fox* (Princeton, N.J.).

Ganea, Tudor. Zur Charakterisierung einparametriger topologischer Gruppen. Com. Acad. R. P. Române **1**, 731–732 (1951). (Romanian. Russian and German summaries)

Attention is drawn to the fact that the statement of an exercise in Bourbaki's treatise [Topologie générale, ch. V–VIII, Actualités Sci. Ind., no. 1029, Hermann, Paris, 1947, p. 16, exerc. 4; MR **9**, 261, 735] is incorrect. The outline of the proof given in that exercise is entirely correct but what it actually proves is that the group G (connected, and with the complement of the identity not connected) is algebraically isomorphic to the group R of the real line, and has a finer topology (and not, as wrongly stated, that G and R are topologically isomorphic). The author points out that the topology of G may in fact be strictly finer than that of R, and that additional conditions can be introduced to restore topological isomorphism, for instance local compactness or local connectedness of G.

J. Dieudonné (Evanston, Ill.).

Obreanu, Filip. Topologies compatibles avec le groupe additif de la droite réelle. Com. Acad. R. P. Române **1**, 557–558 (1951). (Romanian. Russian and French summaries)

The author comments on the well-known fact that algebraically the additive group R of the real line is isomorphic to any product R^n, hence there are topologies on R, compatible with the group structure, and giving a topological group isomorphic to R^n.

J. Dieudonné (Evanston, Ill.).

See also: Bartle, p. 62.

Lie Groups, Lie Algebras

Harish-Chandra. Integrable and square-integrable representations of a semisimple Lie group. Proc. Nat. Acad. Sci. U.S.A. **41**, 314–317 (1955).

Let G be a connected semi-simple Lie group with a finite center and π an irreducible unitary representation of G on a Hilbert space H. The author calls π integrable (square-integrable) if there exists an element ψ of H such that the inner product $(\psi, \pi(x)\psi)$ is an integrable (square-integrable) function of x on G. For square integrable π the author defines a positive constant d_π by

$$(d_\pi)^{-1}=\int_G (\psi, \pi(x)\psi)^2 dx,$$

where dx refers to Haar measure on G and ψ is any unit vector in H. From the analogue of the Schur orthogonality relations the following expression of the character T_π of a square-integrable π is deduced:

$$T_\pi(f)=d_\pi\int dx\int f(xyx^{-1})(\phi,\pi(y)\phi)dy,$$

where ϕ is any unit vector in H and $f(x)$ any indefinitely differentiable function of compact support on G. Now assume that G is simple, non-compact, that its first Betti number is one and a positivity condition on certain roots of G. Then an expression for d_π is obtained which is quite analogous to Weyl's formula for the degrees of the finite-dimensional representations of the semi-simple Lie groups. This gives an explicit expression for the contribution of the square-integrable irreducible unitary representation to the Plancherel formula on G. *F. I. Mautner.*

Naĭmark, M. A. On the description of all unitary representations of the complex classical groups. II. Mat. Sb. N.S. 37(79), 121–140 (1955). (Russian)

The results of Part I of this paper [Mat. Sb. N.S. 35(77), 317–356 (1954), MR 16, 567] are used to derive the following basic theorem. The contraction of an irreducible unitary representation of a complex classical (simple) Lie group to the maximal compact subgroup decomposes into irreducible representations of the subgroup, among which one of maximum highest weight occurs exactly once. From this follows the form of the general elementary positive linear functional on a ring associated with the group, the kernel being given by an exponential function on a maximal abelian subgroup involving certain constants ϱ_j which are either real or occur in complex conjugate pairs. These elementary positive definite functionals are in mutual correspondence with the irreducible representations. Conditions are given in terms of the ϱ_j for insuring that the associated representation be in the principal or complementary series of representations respectively.

I. E. Segal (Chicago, Ill.).

van Est, W. T. On the algebraic cohomology concepts in Lie groups. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17, 225–233, 286–294 (1955).

Continuing his investigation of the relations among the various cohomology groups that can be attached to a Lie group [same Proc. 56, 484–492, 493–504 (1953); MR 15, 505], the author here obtains the following results. Let M be a connected Lie group, F a compact subgroup of M, V a representation space for M. Then the group-cohomology group for M in V, computed from the complex of equivariant homogeneous infinitely differentiable group cochains, is isomorphic with the cohomology group for M/F in V, computed from the complex of those of the above group cochains whose values depend only on the cosets mod F of their arguments. If F is a maximal compact subgroup of M these cohomology groups are also isomorphic with the cohomology group of the complex of relative cochains for the Lie algebra of M, relative to the Lie algebra of F, in V.

It is then deduced from previously known results that, if furthermore the Lie algebra of F is totally non-homologous to 0 in the Lie algebra of M, the cohomology group of the Lie algebra of M in V is isomorphic with the tensor product of the topological cohomology group (over the reals) of M and the above group-cohomology group for M in V.

If A is an arbitrary connected abelian Lie group on which M operates, and if A' is the universal covering group of A, then A' may be regarded as a representation space for M in a natural way. It is shown that the group-cohomology groups $H^i(M, A')$ and $H^i(M, A)$ are isomorphic for $i\geq 3$, in the general case, and are isomorphic for $i\geq 1$, in the case where M is simply connected.

G. Hochschild (Berkeley, Calif.).

Lavendhomme, R. Les algèbres de Lie de caractéristique quelconque. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 133–139 (1954).

In this article an attempt is made to show that (1) every solvable linear Lie algebra, finite-dimensional over an algebraically closed field, has a proper vector, and (2) every semi-simple finite-dimensional Lie algebra is a direct sum of simple algebras. A counter example to each of these assertions has been given for characteristic $p\neq 0$.

W. G. Lister (Providence, R.I.).

Hochschild, G. Simple algebras with purely inseparable splitting fields of exponent 1. Trans. Amer. Math. Soc. 79, 477–489 (1955).

Si K est une extension galoisienne d'un corps C, on sait que le sous-groupe du groupe de Brauer des classes d'algèbres simples de centre C, formé des classes d'algèbres dont K est corps de décomposition, est isomorphe au groupe des classes d'équivalence des extensions du groupe multiplicatif K^* par le groupe de Galois de K sur C. L'auteur montre comment sa récente théorie de la cohomologie des algèbres de Lie "restreintes" [Amer. J. Math. 76, 555–580 (1954); MR 16, 109] permet d'obtenir des résultats correspondant aux précédents lorsque K est une extension purement inséparable d'exposant 1 de C (supposé de caractéristique $p>0$). Soit A une algèbre simple de centre C ayant K comme sous-corps commutatif maximal; pour tout $s\in A$, soit $D_s(a)=sa-as$, et soit S l'ensemble des $s\in A$ tels que $D_s(K)\subset K$; pour $s\in S$, soit $\varphi(s)$ la restriction de D_s à K. S est évidemment une algèbre de Lie restreinte sur C; il résulte d'un théorème de Jacobson (dont l'auteur donne une démonstration simple) que φ applique S sur l'algèbre de Lie restreinte T des dérivations de K s'annulant dans C; comme K est le noyau de φ, (S, φ) est une extension de K par T. En outre, S et T ont toutes deux une structure d'espace vectoriel sur K, φ est linéaire pour ces structures, et on a les deux conditions suivantes: (i) $[v_1\varrho_1, v_2\varrho_2]=v_1(\varrho_1\cdot v_2)\varrho_2-v_2(\varrho_2\cdot v_1)\varrho_1+v_1v_2[\varrho_1,\varrho_2]$ pour v_1, v_2 dans K, ϱ_1, ϱ_2 dans S ($\sigma\cdot v$ est le transformé de $v\in K$ par $\sigma\in S$); (ii) $(v\varrho)^p=v^p\varrho^p+t_{\varrho}^{p-1}(v)$, où $v\in K$, $\varrho\in S$ et $t_\sigma(v)=\sigma\cdot v$ pour tout $\sigma\in S$.

Une extension (S, φ) de K par T pour laquelle il existe sur S une structure de K-espace vectoriel pour laquelle (i) et (ii) sont vérifiées, et φ est K-linéaire, est dite régulière. L'auteur montre d'abord que les structures de K-espace vectoriel sur S ayant ces propriétés sont nécessairement transformées l'une de l'autre par un automorphisme d'extension. Il prouve ensuite son théorème principal, affirmant qu'à toute extension régulière (S, φ) de T correspond une algèbre simple V_s de centre C ayant K pour sous-corps commutatif maximal; il y parvient en construisant directement V_s par une méthode ingénieuse et assez compliquée. Il est ensuite assez facile de prouver, par des méthodes analogues à celles de la théorie classique des „produits croisés" que la correspondance ainsi établie entre extensions régulières de T et classes d'algèbres simples dans le groupe de Brauer, est un isomorphisme de groupe. L'auteur indique d'ailleurs pour terminer comment on peut expliciter cet isomorphisme quand on a choisi une base de T sur K. *J. Dieudonné.*

Topological Vector Spaces

Landsberg, Max. Lokalkonvexe Räume von Grade $r(0<r\leq 1)$. Wiss. Z. Tech. Hochsch. Dresden **2**, 369–372 (1953).

E sei ein komplexer linearer Raum. Eine Teilmenge M heißt konvex vom Grad $r(0<r\leq 1)$, wenn für $a, b>0$ und $a^r+b^r=1$ stets $aM+bM\subset M$ gilt. Ist stets auch $e^{i\varphi}M\subset M$, so heißt M absolutkonvex vom Grad r. Ein topologischer linearer Raum E heißt lokalkonvex vom Grad r, wenn er ein fundamentales Nullumgebungssystem aus absolutkonvexen Mengen vom Grad r besitzt. Diese Nullumgebungen lassen sich stets durch Halbnormen $N(x)$ vom Grad r charakterisieren, für die $N(\alpha x)=|\alpha|^r N(x)$ neben $N(x+y)\leq N(x)+N(y)$ gilt. Es wird untersucht, welche Bedingungen ein System von Halbnormen vom Grad r erfüllt, das zu einem lokalkonvexen Raum r gehört, und wann man mit einer einzigen Norm vom Grad r auskommt. Jeder lokalkonvexe Raum vom Grad r ist auch lokalkonvex vom Grad $s<r$, aber nicht umgekehrt, speziell also im allgemeinen kein lokalkonvexer Raum im üblichen Sinn. *G. Köthe* (Zbl **51**, 85).

Choquet, Gustave. Sur le théorème des points-selle de la théorie des jeux. Bull. Sci. Math. (2) **79**, 48–53 (1955).

Another and rather different proof of the fundamental theorem of 2-person zero-sum games is given. In its general context the theorem reads: If X and Y are compact convex subsets of the topological vector spaces (over the reals) L_x, resp. L_y, and if $f(x, y)$ is a continuous bilinear functional on $L_x\times L_y$, then: (1) if $f(a, y)$ and $f(x, b)$ are not identically 0 for any $a\in X$ or $b\in Y$ and if X and Y are strictly convex, then there is a unique minimax point for f in $X\times Y$; (2) in the general case, uniqueness may fail, existence remains. Via several simple variable changes, (1) of the theorem is reduced to the following lemma: In a topological vector space L, let φ and ψ be two linear continuous functionals, $\varphi\not\equiv 0$. If $K\subset L$, K compact, $0\in K$ and

$$K-(0)\subset\{x|0<\varphi(x)\},$$

then $\varrho(t)=\min_K[\varphi(x)+t\psi(x)]=0(t)$ (Proof: Let $V=\{x\mid |\varphi|\leq\varepsilon, |\psi|\leq\varepsilon\}$, $K_\varepsilon=K\cap V$, $K_{1-\varepsilon}=\overline{(K-K_\varepsilon)}$, $\varepsilon>0$. On $K_{1-\varepsilon}$, $\varphi>C>0$, some C; $\varphi+t\psi>\frac{1}{2}C$ on $K_{1-\varepsilon}$, t small. Thus $\min_K=\varrho(t)=\min_{K_\varepsilon}$, t small, and $0\geq\min_{K_\varepsilon}\geq -t\varepsilon$; whence $|\varrho(t)|\leq|t|\varepsilon$.) The general result follows by filter approximations for which the first result is valid. *B. Gelbaum* (Minneapolis, Minn.).

Bartle, Robert G. Implicit functions and solutions of equations in groups. Math. Z. **62**, 335–346 (1955).

Some of the theory of implicit functions in Banach space developed by T. H. Hildebrandt and L. M. Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927)] is extended to functions with domain and range in topological groups (not necessarily abelian) with a right invariant metric and a "norm" defined in terms of the metric. A derivative of a function at a point is defined as a continuous homomorphism which approximates the function locally in an appropriate way, and local inversion theorems and an implicit function theorem are obtained by assuming (with other hypotheses) that the derivative is "smooth". (Mapping f such that $f(x_0)=y_0$ is said to be smooth at x_0 if there are positive numbers α and β such that, if $0<\|y-y_0\|<\beta$, the equation $f(x)=y$ has at least one solution x such that $\|x-x_0\|<\alpha\|y-y_0\|$.) Since the derivative is not required to be one-to-one, this extends results of L. M. Graves [Duke Math. J. **17**, 111–114 (1950); MR **11**, 729].

Continuation of solutions is discussed, and conditions given which insure that if $y_0=f(x)$ has k solutions or a continuum of solutions and if y_1 is near y_0, then $y_1=f(x)$ has, respectively, k solutions or a continuum of solutions. *J. Cronin* (Southbridge, Mass.).

Schaefer, Helmut. Positive Transformationen in lokalkonvexen halbgeordneten Vektorräumen. Math. Ann. **129**, 323–329 (1955).

Perron [Math. Ann. **64**, 1–76 (1907)] and Frobenius [S.-B. Preuss. Akad. Wiss. **1908**, 471–476] proved that a non-singular (finite-dimensional) matrix with positive elements has at least one positive eigenvalue. R. Jentzsch [J. Reine Angew. Math. **141**, 235–244 (1912)] proved a corresponding theorem for linear integral equations with a positive kernel, and Alexandroff and Hopf [Topologie, Bd. I, Springer, Berlin, 1935, p. 480] gave a proof of the Perron-Frobenius theorem based on Brouwer's fixed-point theorem. Since then various writers extended this theorem to "positive" completely continuous mappings in Hilbert and Banach spaces including non-linear mappings. The present paper deals with the extension to locally convex linear topological spaces: let E be such a space which in addition is supposed to be complete, separable (i.e. "separé" in the sense of Bourbaki) and partially ordered. It is supposed that the set C of non-negative elements (i.e. of elements $\geq$ the zero element θ of E) contains at least one element $\neq\theta$. A (not necessarily linear) mapping T of E into itself is called positive if it maps C into itself; a positive T is called strictly positive if it maps a subset M of C whose closure does not contain θ in a set of the same property. Finally, a positive T is called p-completely continuous if it maps the intersection of C with a closed neighborhood of θ in a compact set. Then the following assumption is proved to be sufficient for the existence of a positive number λ_0 and positive element x_0 satisfying $\lambda_0 x_0=T(x_0)$: T is strictly positive and p-completely continuous. The proof is based on Tychonoff's fixed-point theorem [Math. Ann. **111**, 767–776 (1935)]. An alternate proof utilizing the resolvent and making no use of the fixed-point theorem is given in the special case that T is linear. *E. H. Rothe* (Ann Arbor, Mich.).

Scarfiello, R. Sur le changement de variables dans les distributions et leurs transformées de Fourier. Nuovo Cimento (9) **12**, 471–482 (1954).

Soit sur $X=R^n$, $\overset{p}{\mathcal{D}}$ l'espace des formes différentielles de degré p, indéfiniment différentiables à support compact; $\overset{n-p}{\mathcal{D}'}$=dual de $\overset{p}{\mathcal{D}}$=courants de degré $(n-p)$ [les topologies sont celles de Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833]. Soit H un isomorphisme de X dans X; si $\overset{0}{\varphi}\in\overset{0}{\mathcal{D}}$, on pose $H'\overset{0}{\varphi}(x)=\overset{0}{\varphi}(Hx)$, et on définit $H\overset{n}{T}$, $\overset{n}{T}\in\overset{n}{\mathcal{D}}$, par transposition. Si $\overset{n}{\varphi}=\varphi dx\in\overset{n}{\mathcal{D}}$, $H'\overset{n}{\varphi}=\varphi(Hx)|H|dx$, $|H|$=module du jacobien de H; on définit $H\overset{0}{T}$, $\overset{0}{T}\in\overset{0}{\mathcal{D}}$, par transposition. De façon générale ($\overset{p}{\varphi}\in\overset{p}{\mathcal{D}}$):

$$H'd\overset{p}{\varphi}=dH'\overset{p}{\varphi},\quad H'(\overset{p}{\varphi}\wedge\overset{q}{\psi})=H'\overset{p}{\varphi}\wedge H'\overset{q}{\psi};$$

on passe aux courants par transposition. Application à la masse de Dirac δ considérée d'abord comme courant de degré 0 puis comme courant de degré n.

Soit maintenant Y le dual de X. Soit $\overset{p}{\mathcal{S}}_x$=espace des formes différentielles sur X de degré p, indéfiniment différentiables, à décroissance rapide (au sens de Schwartz, loc. cit.), muni de la topologie usuelle; $\overset{n-p}{\mathcal{S}}_x'$=espace dual du précédent, i.e. espace des courants tempérés de degré $n-p$. Ecrivons avec l'A. $\overset{p}{\bigwedge} X$ au lieu de $\overset{p}{\bigwedge} X+i\overset{p}{\bigwedge} X$. On a: $\overset{p}{\mathcal{S}}_x=\overset{0}{\mathcal{S}}_x\otimes\overset{p}{\bigwedge} Y$; idem pour $\overset{p}{\mathcal{S}}_x'$. Si donc A est une application linéaire continue de $\overset{0}{\mathcal{S}}_x'$ dans lui même, on définit $A=A\otimes 1$, application linéaire continue de $\overset{p}{\mathcal{S}}_x'$ dans lui même. Soit maintenant B une application linéaire continue de $\overset{n}{\mathcal{S}}_x'$ dans $\overset{0}{\mathcal{S}}_y'$. Alors B définit canoniquement une application linéaire continue de $\overset{n-p}{\mathcal{S}}_x'$ dans $\overset{p}{\mathcal{S}}_y'$ (la structure de $\mathcal{S}$ n'intervient pas); en effet, soit $\overset{p}{\omega}$ la forme identique de degré p sur X et à valeurs dans $\overset{p}{\bigwedge} X$. Si $\overset{n-p}{T}\in\overset{n-p}{\mathcal{S}}_x'$, $\overset{n-p}{T}\wedge\overset{p}{\omega}=\overset{n}{S}$ est dans $\overset{n}{\mathcal{S}}_x'\otimes\overset{p}{\bigwedge} X$, donc $(B\otimes 1)S$ est dans $\overset{0}{\mathcal{S}}_y'\otimes\overset{p}{\bigwedge} X=\overset{p}{\mathcal{S}}_y'$, ce qui définit l'application cherchée. Application au cas où B=transformation de Fourier $\mathcal{F}$ (ici $\mathcal{S}$ intervient).

Cas où H est un isomorphisme sur; soit $\overset{p}{H}$ l'isomorphisme contragrédient. On généralise aux formes de degré quelconque la formule $\mathcal{F}\overset{0}{H}\overset{0}{T}=\overset{n}{H}\mathcal{F}\overset{0}{T}$, $\overset{0}{T}$ de degré 0. Application: $X=R$, $n=1$, H=homothétie de rapport $\lambda>0$ (puis quelconque), $T=\log|x|$, regardé successivement comme courant de degré 0 et 1. *J. L. Lions* (Nancy).

Slowikowski, W. A generalization of the theory of distributions. Bull. Acad. Polon. Sci. Cl. III. **3**, 3–6 (1955).

L'auteur développe un formalisme contenant comme cas particuliers la construction du groupe des nombres rationnels à partir du groupe des nombres entiers (groupe quotient), la construction des opérateurs de Mikusiński [Studia Math. **11**, 41–70 (1950); MR **12**, 189] et la construction des distributions [Schwartz, Théorie des distributions, t. I, Hermann, Paris, 1950; MR **12**, 31], à partir des fonctions continues. *L. Schwartz* (Paris).

Slowikowski, W. On the theory of operator systems. Bull. Acad. Polon. Sci. Cl. III. **3**, 137–142 (1955).

Cet article contient des compléments et exemples au théorème topologique fondamental démontré dans la note analysée ci-dessus. *L. Schwartz* (Paris).

Korevaar, Jacob. Distributions defined from the point of view of applied mathematics. I. Fundamental sequences. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 368–378 (1955).

Définition des distributions par des „suites fondamentales" de fonctions. Définition analogue des distributions à croissance exponentielle (pour la transformation de Laplace). Exemples d'applications des distributions à la mécanique.

L. Schwartz (Paris).

Korevaar, Jacob. Distributions defined by fundamental sequences. II. Derivatives and antiderivatives, Laplace transformation. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 379–389 (1955).

Suite de l'article analysé ci-dessus. *L. Schwartz.*

Banach Spaces, Banach Algebras

★ **Ljusternik, L. A., und Sobolew, W. I. Elemente der Funktionalanalysis.** Akademie-Verlag, Berlin, 1955. xi+256 pp. DM 25.00.

Translation of the authors' Elementy funkcional'nogo analiza [Gostehizdat, Moscow, 1951; MR **14**, 54].

Ritt, R. K. A generalization of inner product. Michigan Math. J. **3**, 23–26 (1955).

If B is a normed linear space and if B^*, the conjugate space of B, is strictly convex, then each point of the unit sphere of B has through it a unique hyperplane of support. [See Day, Trans. Amer. Math. Soc. **78**, 516–528 (1955), for references; MR **16**, 716]. In this case each x in B determines a unique φ_x in B^* such that $\varphi_x(x)=1=\|x\|\cdot\|\varphi_x\|$. This paper shows first that if B^* is uniformly convex then φ_x depends continuously on x, and then shows that in a space B with strictly convex conjugate space the following three conditions are equivalent: (a) If $\|x\|=\|y\|=1$, then $\varphi_x(y)=\varphi_y(x)$. (b) If x and $y\in B$, then $\|x\|^2\varphi_x(y)=\|y\|^2\varphi_y(x)$. (c) If $\langle x, y\rangle=\|x\|^2\varphi_x(y)$, then $\langle\ ,\ \rangle$ is an inner product in B. *M. M. Day* (Urbana, Ill.).

Marinescu, G. Sur le théorème de séparation des ensembles convexes par un hyperplan. Com. Acad. R. P. Romîne **3**, 301–303 (1953). (Romanian. Russian and French summaries)

This note contains a proof of the lemma of Leibenzon [Uspehi Mat. Nauk (N.S.) **7**, no. 2 (48), 165–167 (1952); MR **13**, 848] that two disjoint convex sets in a linear space E are respectively contained in disjoint convex sets whose union is E. *M. M. Day* (Urbana, Ill.).

Stein, P., and Peck, J. E. L. The differentiability of the norm of a linear operator. J. London Math. Soc. **30**, 496–501 (1955).

Let E be a normed complex-linear space, and let $A(t)$ be a function from a real interval into E with a derivative A_0' at t_0. Then it is proved that $|A|$ has right and left derivatives at t_0 and that these are dominated by $|A_0'|$. If E is a Banach algebra with identity and each $A(t)$ is also non-singular, then also

$$\limsup_{t\to t_0}\ \big|\,|A^{-1}|^{-1}-|A_0^{-1}|\,\big|/|t-t_0|\leq|A_0'|.$$

This includes a result of Ostrowski [C. R. Acad. Sci. Paris **231**, 1019–1021 (1950); MR **12**, 396] in which E is n^2-dimensional Euclidean space and A an n-by-n matrix. The final theorem of this note asserts that if A is a differentiable matrix function of t and if the maximum root of A^*A has multiplicity 1, then $d|A|^2/dt$=twice the real part of $(Ax, A'x)$, where $|x|=1$ and $|Ax|=|A|$. *M. M. Day.*

Ivanov, N. A. On Gateaux and Fréchet differentials. Uspehi Mat. Nauk **10**, no. 2(64), 161–166 (1955). (Russian)

This note gives some conditions under which a Gateaux differential of a real-valued function on a normed linear space (usually taken finite-dimensional) is a Fréchet

differential. The extra hypotheses involve Lipschitz conditions or second differentials. Examples in two-dimensional space illustrate the results. *M. M. Day.*

Phillips, R. S. The adjoint semi-group. Pacific J. Math. **5**, 269–283 (1955).

Let the semi-group T_s, $s\geq 0$, of bounded linear transformations in a Banach space X satisfy the conditions: $T_0=I$, $T_s x$ is strongly continuous for $s>0$ and $\int_0^1 \|T_s x\|ds<\infty$. T_s is said to be of class $(0, A)$, $(0, C)$, or (C) if it satisfies $\lim_{\lambda\to\infty} \lambda\int_0^\infty \exp(-\lambda s)T_s x\,ds=x$, $\lim_{s\downarrow 0} s^{-1}\int_0^s T_s x\,ds=x$, or $\lim_{s\downarrow 0} T_s x=x$, respectively. It is said to be of class $(0, A)^*$, $(0, C)^*$, or $(C)^*$ if it is of class $(0, A)$, $(0, C)$, (C), respectively, and if in addition $\|T_s^* x^*\|$, $0\leq s\leq 1$, is majorized by integrable function for each $x^*\in X^*$ (=the adjoint space of X; B^* denotes the adjoint operator of B). Let A be the infinitesimal generator of T_s and let X^+ be the closure of the domain of A^*. The "adjoint semi-group" T_s^+ of T_s is defined as the restriction of T_s^* to X^+. It is proved (th. 2.1) that, if T_s is of class $(0, A)^*$, $(0, C)^*$, or $(C)^*$, then the adjoint semi-group T_s^+ is of class $(0, A)$, $(0, C)$, or (C), respectively, and the smallest closed extension of the infinitesimal generator of T_s^+ is the largest restriction of A^* with domain and range in X^+. *K. Yosida* (Tokyo).

Fenyő, István. Méthode de L. V. Kantorovich pour la solution des équations nonlinéaires considérées dans des espaces abstraits. Mat. Lapok **3**, 11–46 (1952). (Hungarian. Russian and French summaries)

Simplified presentation of Kantorovich's method for solving nonlinear operator equations. Following Marx, Fréchet and Kantorovich, the author defines the first and second differential of a nonlinear operator in Banach space, and sets up a generalized Newton iteration scheme. Applications are given. *P. D. Lax* (New York, N.Y.).

Pol'skiĭ, N. I. On the convergence of certain approximate methods of analysis. Ukrain. Mat. Ž. **7**, 56–70 (1955). (Russian)

Approximative theorems relating to the spectral values and eigenvectors of a completely continuous linear operator A are obtained. Their validity depends upon the existence in E (the Banach space where A operates) of a basis $\{\varphi_n\}$. If Φ_n are the associated biorthogonal linear functionals, the projections P_n are defined by:

$$P_n(x)=\sum_{i=1}^n \Phi_i(x)\varphi_i,\quad x\in E.$$

Clearly

$$\|A-P_nA\|+\|A-AP_n\|\to 0 \text{ as } n\to\infty.$$

The operators P_nA are thus approximants of A. The Galerkin approximant to the equation $x-\lambda A_n=f$ is the equation $x_n=\sum_{i=1}^n c_i^{(n)}\varphi_i$, where $c_i^{(n)}$ are determined by (*) $\Phi_i(x_n-\lambda Ax_n-f)=0$, $i=1, 2, \cdots, n$. Theorem 1: If λ is not an eigenvalue of A, then there is an n for which the Galerkin equation has a solution with unicity insured by (*). Finally, $|x-x_n|\leq(1+\varepsilon_n)|x-P_nx|$, where $\varepsilon_n\to 0$ as $n\to\infty$. Theorem 2: The set of eigen values of A is identical with the set of limit points of the set of all eigen values of the approximants P_nA. Theorem 3: Let λ_n be a set of approximating eigen values which converge to an eigen value λ_0 of A; from the (normed) approximating eigen vectors f_n associated with λ_n, one can choose a subsequence converging strongly to an eigen vector of A. Furthermore, if any subsequence of the f_n converges to an eigenvector f of A, then the associated eigenvalue is λ_0. Similar results are valid for invariant subspaces associated with λ_0 and the λ_n, and for the operators $P_n-\lambda_nP_nA$. Applications are given to the study of the equation $x-A(\lambda)x=f$, where $A(\lambda)$ is a completely continuous operator depending analytically on the parameter λ. More generally the equation $Ax-B(\lambda)x=f$ is considered. *B. Gelbaum.*

Kato, Tosio. Linear differential equations in Banach spaces. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-11**, i+11 pp. (1955).

This paper is concerned with the solution of the following abstract Cauchy problem: $du/dt=A(t)u(t)+f(t)$ $(a\leq t\leq b)$, $u(a)$ given, where $u(t)$ and $f(t)$ take values in a Banach space and the $A(t)$ form a family of unbounded semi-group generators. An operator A is said to be of class (S') if there exists a bounded linear operator Q with bounded inverse such that $\bar{A}=QAQ^{-1}$ satisfies the Hille-Yosida criterion for the generation of a semi-group of operators. Given a family of operators $A(t)$, $a\leq t\leq b$, of class (S') with corresponding $Q(t)$, the author assumes that the operator-valued function $Q(t)$ is strongly continuously differentiable in t and shows that there is at most one solution to the above Cauchy problem. In order to establish the existence of a solution, he further assumes (i) that there exists an operator-valued function $R(t)$ of bounded regular operators which is twice strongly continuously differentiable and such that

$$(\bar{A}t)=R(t)A(t)R(t)^{-1}$$

has a domain independent of t, (ii) that

$$B(t)=[I-\bar{A}(t)][I-\bar{A}(t)]^{-1}$$

is strongly continuously differentiable, and (iii) that $f(t)$ is strongly continuously differentiable. The proofs are merely indicated, but follow the general method developed by Kato in an earlier but less general paper [J. Math. Soc. Japan **5**, 208–234 (1953); MR **15**, 437]. Although the present results are considerably deeper than those obtained by the reviewer [Trans. Amer. Math. Soc. **74**, 199–221 (1953); MR **14**, 882], they still do not contain them. *R. S. Phillips* (Los Angeles, Calif.).

Dunford, Nelson, and Schwartz, J. Convergence almost everywhere of operator averages. Proc. Nat. Acad. Sci. U.S.A. **41**, 229–231 (1955).

L_p $(1\leq p\leq\infty)$ denoting the usual Lebesgue space associated with a measure-space (S, Σ, μ), operators T are considered having domain $D(T)$ containing $L_1\cap L_\infty$ and range $R(T)$ contained in $L_1\cup L_\infty$ and for which

$$|T|_p=\sup\{|Tf|_p: f\in D(T),\ |f|_p\leq 1\}$$

is finite. Two theorems are given about k-parameter, strongly measurable semigroups of operators in L_1, each asserting that under suitable conditions the parameter-averages tend a.e. to limits. As a specimen we quote Theorem 2: Let $\{T(t_1, \cdots, t_k)\}$ $(t_1>0, \cdots, t_k>0)$ be the semigroup and suppose that $|T(t_1, \cdots, t_k)|_p\leq 1$ for $p=1, \infty$ and all t's just specified. For each $\alpha>0$ and each $f\in L_1$,

$$A(\alpha)f=\alpha^{-k}\int_0^\alpha\cdots\int_0^\alpha T(t_1, \cdots, t_k)f\,dt_1\cdots dt_k$$

has the property that $\lim_{\alpha\to\infty} A(\alpha)f$ exists a.e.

Some indications of the proof are given. These involve a two-fold reduction of the theorems to their analogues for discrete semigroups and for the case of positive operators, followed by reference to earlier work [N. Dunford, Acta Sci. Math. Szeged **14**, 1–4 (1951); MR **13**, 49; E. Hopf, J. Rational Mech. Anal. **3**, 13–45 (1954); MR **15**, 636]. *R. E. Edwards* (London).

Lorentz, G. G. Majorants in spaces of integrable functions. Amer. J. Math. **77**, 484–492 (1955).

Let C be a class of integrable functions on [0, 1] such that (a) for some $\varepsilon>0$ there is an $f \in C$ such that $f \geq \varepsilon$, (b) if $f \in C$ and if g and f are equimeasurable, then $g \in C$. Let $X(C)=\{h \mid \|h\|=\sup_{c \in C} \int_0^1 |h|\, c(x)dx<\infty\}$; X' (the dual, as opposed to the conjugate space) $=\{g \mid g$ integrable, $\int_0^1 hg\,dx$ finite for all $h \in X(C)\}$. The author discusses the problem of the Hardy-Littlewood Property (HLP) of $X(C)$, namely, necessary and sufficient conditions (on C) for the validity of: $f \in X(C)$ implies $\theta(x)=\sup_y \int_x^y f(t)(y-x)^{-1}dt \in X(C)$. Results: $X(C) \in$ HLP if and only if there is a constant A such that $\|\tilde{c}\|_{X'} \leq A$ for all $c \in C$, where $\tilde{c}(x)=\int_x^1 [c(t)/t]dt$. Let $\varphi \geq 0$, and decreasing on [0, 1], $\Phi(x)=\int_0^x \varphi(t)\,dt$. Let $\Lambda(\varphi)=\{f \mid \|f\|_\Lambda=\sup_r \int_0^1 \varphi_r(x)|f(x)|dx<\infty\}$, where φ_r is any function which is equimeasurable with φ. Let $M(\varphi)=\{f \mid \|f\|_M=\sup_a (1/\Phi(a)) \int_0^a f^*(t)dt<\infty\}$ where f^* is the decreasing equimeasurable associate of f. Theorem: $\Lambda(\varphi) \in$ HLP if and only if $\limsup_{a \to 0} \Phi(2a)/\Phi(a)<2$. $M(\varphi) \in$ HLP if and only if there is a constant A such that $\int_0^a \Phi(x)x^{-1}dx \leq A\Phi(a)$ for $0 \leq a \leq 1$. Finally, the Orlicz space $L_\Phi \in$ HLP if and only if the conjugate function Ψ satisfies $\Psi(2b) \leq M\Psi(b)$ for all $b \geq$ some b_0 and some M.

B. Gelbaum (Minneapolis, Minn.).

Marcus, S. Quelques ensembles rares dans certains espaces fonctionnels. Com. Acad. R. P. Romîne **5**, 291–293 (1955). (Romanian. Russian and French summaries)

The author remarks that certain function spaces are nowhere dense in certain others, where all are subspaces of bounded real functions with least-upper-bound norm.

M. M. Day (Urbana, Ill.).

Singer, Ivan. Contributions à la théorie des lattices vectorielles normées (*KB*-linéaux). Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **6**, 79–134 (1955). (Romanian. Russian and French summaries)

The author considers vector lattices in which the norm is monotone; that is, $|x| \leq y$ implies $\|x\| \leq \|y\|$. A set F of non-negative linear functionals of norm one on such a space E is called a normed positive-total (ptn) set if $x \geq 0$ whenever $f(x) \geq 0$ for all f in F. Observing that computations with F generalize the notion of "evaluation at a point" available in special function spaces, such as m or C, the author defines a quasiunit (unit) in E to be an element e of E such that $f(e)=1$ for each f in F (and also $\|e\|=1$). It is proved that e is a unit in E if and only if it is a unit in the sense of Kakutani [Ann. of Math. (2) **42**, 523–537 (1941); MR **2**, 318], and if and only if $\|e\|=1$ and every x within distance 1 of e is nonnegative, and if and only if $f(e)=1$ for every positive f of norm one in L^*. Then every E with a unit is an abstract M-space [Kakutani, ibid. **42**, 994–1024 (1941); MR **3**, 205], and the positive face of the unit sphere in L^* is the w^*-closed convex hull of F.

It is shown that e is a quasiunit if and only if

$$\inf \{u \mid \|u-x\| \leq 1 \text{ implies } x \geq 0\}$$

exists; then e is that inf. This shows that the quasiunit determined by a ptn F is independent of F, as long as the quasiunit exists for F at all. Several definitions of units are discussed. The last section applies some of the earlier discussion to normed rings.

M. M. Day (Urbana, Ill.).

Marinescu, G. La méthode des approximations successives en groupes à norme abstraite. Com. Acad. R. P. Române **1**, 547–550 (1951). (Romanian. Russian and French summaries)

By dropping the parts of the structure involving scalar multiplication the author shows that results of Kantorovič, Vulih, and Pinsker [Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 340] on the method of successive approximation can be carried out in certain groups. *M. M. Day.*

Hilbert Space

Naĭmark, M. A., and Fomin, S. V. Continuous direct sums of Hilbert spaces and some of their applications. Uspehi Mat. Nauk **10**, no. 2(64), 111–142 (1955). (Russian)

This is an expository account of certain basic aspects of the subject described in the title. There are seven sections. Section one is introductory and contains references to a lecture of Kolmogorov in Moscow in 1944 and a 1948 dissertation of Adel'son-Velskiĭ in which parts of the theory of continuous sums were worked out in advance of the appearance of von Neumann's fundamental paper in 1949 [Ann. of Math. (2) **50**, 401–485 (1949); MR **10**, 548]. Apparently the only part of this work which is available in published form is a brief note of Adel'son-Velskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **67**, 957–959 (1949); MR **11**, 115.] In this connection it should perhaps be mentioned that, except for minor changes, von Neumann's paper was actually completed in 1938. Section two contains the definition of continuous direct sum for the case in which all component Hilbert spaces of the same dimension have been identified in advance so that one can consider L^2 functions from a measure space to a fixed Hilbert space. Section three is devoted mainly to a proof of the theorem setting up a one-to-one correspondence between direct sum decompositions of a Hilbert space and commutative weakly closed subalgebras of the algebra of all of its bounded linear operators. It also includes a treatment of the isomorphism between the algebra of all operators decomposed by the given decomposition of the Hilbert space and the commutator of the commutative algebra associated with the decomposition. The main theorem of section 4 asserts that if $\mathcal{A}$ is a family of decomposable operators $\lambda \to A_\lambda$ (here λ varies in the measure space associated with the decomposition and A_λ is an operator in the λth Hilbert space) and if the associated commutative algebra is maximal amongst commutative subalgebras of the commutator $\mathcal{A}'$ of $\mathcal{A}$ then for almost all λ the family of operators A_λ is an irreducible family in H_λ. The authors also write down an argument designed to establish the converse of this result. There is a gap at one point however and a simple example due to J. M. Cook shows that the converse is in fact false. In section five the main result of the preceding section is applied to prove that a unitary representation of a separable locally compact group is a continuous direct sum of irreducible ones. However this result is stated in a weaker form than that which is now known to be true [see, e.g., Mautner's paper, Trans. Amer. Math. Soc. **78**, 371–384 (1955) MR **16**, 692; and Godement's review of another paper of Mautner MR **13**, 11]. In section six the main result of section four is used to

prove a theorem on the decomposition of measures into ergodic parts. In section seven the theory of continuous direct sums is applied to obtain the Ambrose-Rohlin structure theorem for commutative H-systems=unitary rings. *G. W. Mackey* (Cambridge, Mass.).

Vaĭnberg, M. N. Potential operators and the variational theory of nonlinear operator equations. Uspehi Mat. Nauk (N.S.) **10**, no. 3(65), 223–227 (1955). (Russian)

A summary of the author's doctoral dissertation. The results appeared earlier in Dokl. Akad. Nauk SSSR (N.S.) **75**, 609–612 (1950); **80**, 309–312 (1952); **92**, 457–460 (1953); Uspehi Mat. Nauk (N.S.) **7**, no. 4 (50), 55–102 (1952); **9**, no. 2 (60), 105–112 (1954); Trudy Moskov Mat. Obšč. **3**, 375–406 (1954) [MR **12**, 713; **13**, 353; **15**, 536; **14**, 384; **15**, 967, **16**, 374].

Hongo, Eishi, and Orihara, Masae. A remark on a quasi-unitary algebra. Yokohama Math. J. **2**, 69–72 (1954).

In a recent paper [Comment. Math. Helv. **26**, 275–322 (1953); MR **14**, 660], Dixmier introduced the concept of a quasi-unitary algebra, generalizing the rings of operators of Murray and von Neumann [Ann. of Math. (2) **37**, 116–229 (1936)]. The present paper strengthens a theorem of Dixmier giving conditions for the right and left rings of the algebra to be semi-finite (for terminology see the paper of Dixmier). The authors prove the following theorem: The necessary and sufficient condition that R^d, R^g have no purely infinite component is that there exist self-adjoint, positive, invertible operators M, M' belonging to R^g, R^d respectively, such that $M'=SMS$ and J is the smallest closed extension of $M^{-1}VM'$, for V isometric from the range of M to that of $M'J^{-1}$.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

Griffin, Ernest L., Jr. Some contributions to the theory of rings of operators. II. Trans. Amer. Math. Soc. **79**, 389–400 (1955).

This is a continuation of part I [same Trans. **75**, 471–504 (1953); MR **15**, 539]. The author plans to extend various results of Murray and von Neumann on rings of operators in Hilbert space. In the present paper the notion of coupling operator is extended to general rings of operators, including those of type III, and properties of this coupling are studied such as its invariance. The results are applied to problems about the topology of rings of operators. It is shown that the strongest and σ-weak topologies are purely algebraic and so is the notion of a subring. *F. I. Mautner* (Princeton, N.J.).

Ogasawara, Tôzirô, and Yoshinaga, Kyôichi. A non-commutative theory of integration for operators. J. Sci. Hiroshima Univ. Ser. A. **18**, 311–347 (1955).

The basic result is that if two rings of operators on Hilbert space are $*$-isomorphic, then the isomorphism can be extended to the associated rings of measurable operators [cf. Segal, Ann. of Math. (2) **57**, 401–457; **58**, 595–596 (1953); MR **14**, 991; **15**, 204]. Several other results, that are essentially known, are derived in new ways, with a view to developing the theory of integration of operators for the case of standard rings (or equivalently, in terms of Hilbert algebras), and then extending to general rings by means of the result cited. Notable among these is a theorem of Griffin [Trans. Amer. Math. Soc. **75**, 471–504 (1953); MR **15**, 539] to the effect that certain topologies on standard rings coincide, which is proved here by extending a device known in the theory of commutative rings. The integrable operators can then be introduced for a Hilbert algebra as the products of two square-integrable operators, the latter being essentially given. A number of applications, some of which are new, are given. *I. E. Segal.*

TOPOLOGY

★ **Kuratowski, Kazimierz. Wstęp do teorii mnogości i topologii.** [**Introduction to the theory of sets and topology.**] Państwowe Wydawnictwo Naukowe, Warzawa, 1955. 218 pp. zł. 14.20.

A clear and well written elementary university text. Table of contents: I. Propositional calculus; II. Algebra of sets; III. Functional calculus, cartesian product; IV. Functions, infinite operations; V. Cardinal numbers, countability; VI. Operations on cardinals; VII. Order relations; VIII. Well ordering; IX. Metric spaces; X. Limits of sequences, closure; XI. Various types of sets; XII. Continuous mappings; XIII. Separability; XIV. Completeness; XV. Compactness; XVI. Connectedness; XVII. Continua; XVIII. Local connectedness; XIX. Dimension; XX. The simplex; XXI. Complexes, chains, homologies; XXII. Separation of the plane. *S. Eilenberg.*

Zaric'kiĭ, M. O. Boolean algebra with closure and Boolean algebra with derivation. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 3–6 (1955). (Ukrainian. Russian summary)

The author outlines the axioms of Kuratowski [Fund. Math. **3**, 182–199 (1922)] for the closure A^r of a set A, the inclusion relations between the sets obtained from a given set by iterations of closure and complementation, and similarly axioms and inclusions for the derived set A^d. Some identities are noted; in particular, it is proved that $A^{rcrcr}=A^{dcd}$ where c denotes complementation, but $A^{crc}\neq A^{cdc}$ in general. It is noted that $A^r=A+A^d$, whereas (by a counterexample) the derived set cannot be similarly expressed in terms of A and A^r with elementary operations. Hence, the author argues, in general the derived set is a better basic concept than closure. Some other advantages and disadvantages of each are noted.

P. M. Whitman (Silver Spring, Md.).

Krishnan, V. S. A note on semi uniform spaces. J. Madras Univ. Sect. B. **25**, 123–124 (1955).

The author considers neighborhood structures and generalized uniformities (which are weaker than topological and uniform spaces), and shows that three conditions on these are equivalent. *E. Michael.*

Banaschewski, Bernhard. Über zwei Extremaleigenschaften topologischer Räume. Math. Nachr. **13**, 141–150 (1955).

Let $\alpha, \beta, \cdots, \zeta$, be the classes of Hausdorff, semiregular, regular, 0-dimensional, locally compact, completely regular topological spaces, respectively. For $\tau=\alpha, \cdots, \zeta$, call a τ-space τ-complete if it is not a proper dense subset of some other τ-space; and call it τ-minimal if no proper coarsening of the topology leaves it a τ-space. It is shown that for each of these τ, τ-minimal implies τ-complete. Except for $\tau=\alpha, \gamma$, the converse is also established; and the limitation as regards $\tau=\alpha$ is justified by an example. Concerning the refinements of δ- [or of ζ-] spaces, it is

shown that between two comparable δ- [or ζ-] topologies on a set there can be interpolated a well ordered maximal system of successively coarser δ- [respectively, ζ-] topologies. *R. Arens* (Los Angeles, Calif.).

Bruns, Günter, und Schmidt, Jürgen. Zur Äquivalenz von Moore-Smith-Folgen und Filtern. Math. Nachr. **13**, 169–186 (1955).

That filters and Moore-Smith limits are "equivalent" in the sense that each is adequate to determine a topology has been known since papers of H. Cartan [C. R. Acad. Sci. Paris **205**, 595–598, 777–779 (1937)] and G. Birkhoff [Ann. of Math. (2) **38**, 39–56 (1937)]. The present authors begin by showing that there is a set-theoretical connection between the two. Each Moore-Smith sequence [a net in the sense of Kelley, General topology, Van Nostrand, New York, 1955; MR **16**, 1136] $(\varphi, \geq)$ determines a filter with filterbasis of its residual sets $S_i=\varphi(j), j>i$, and that each filter can be represented in this way by a net defined on a directed system. The discussion then proceeds to discussion of properties of filters which can be described in terms of their representative nets. *M. M. Day.*

Ward, A. J. On relations between certain intrinsic topologies in partially ordered sets. Proc. Cambridge Philos. Soc. **51**, 254–261 (1955).

If $\mathcal{F}$ is a filter over a partially ordered set P, an element x_0 of P is called inferior to $\mathcal{F}$ if, for some set A in $\mathcal{F}$, $x_0 \leq x$ for all x in A. Let $L(\mathcal{F})$ be the set of all inferior elements. We say that o-lim inf $\mathcal{F}=a$ if $L(\mathcal{F})$ is non-empty and sup $L(\mathcal{F})$ is defined and equals a. The concepts of superior element to $\mathcal{F}$ and o-lim sup $\mathcal{F}$ are defined dually. An element a is called medial for $\mathcal{F}$ if $x \leq a \leq y$ for every inferior x and every superior y. If o-lim inf $\mathcal{F}=$ o-lim sup $\mathcal{F}=a$ we say that o-lim $\mathcal{F}=a$. We put so^*-lim $\mathcal{F}=a$ if every refinement $\mathcal{F}_1$ of $\mathcal{F}$ has a further refinement $\mathcal{F}_2$ such that o-lim $\mathcal{F}_2=a$.

The interval topology of a partially ordered set P is defined by taking the closed intervals as a sub-base for the closed sets. The ideal topology is defined by taking the completely irreducible ideals and dual ideals of P as forming a sub-base for the open sets. The filter $\mathcal{F}$ I-converges to a if every open set with respect to the interval topology containing a belongs to the filter $\mathcal{F}$.

Some results: The interval topology of a partially ordered set P is Hausdorffian if and only if there exists at most one medial element for each ultrafilter over P. The interval topology of a lattice is Hausdorffian if and only if I-convergence is equivalent to so^*-convergence. This may be regarded as a solution of Birkhoff's problem 23 or 25 [Lattice theory, Amer. Math. Soc. Colloq. Publ. vol. 25, rev. ed., New York, 1948, p. 62, 64; **10**, 673]. The interval topology of a complete lattice is Hausdorffian if and only if every ultrafilter over L is o-convergent. The ideal topology of a Boolean algebra is Hausdorffian. Some examples are given in which the ideal and o-topologies are compared. *M. Novotný* (Brno).

Nagami, Keiô. Alexandroff's mapping theorem for paracompact spaces. Kôdai Math. Sem. Rep. **7**, 21–22 (1955).

The author proves: A Hausdorff space Y is paracompact if and only if for each open covering $\{U\}$ there is a CW polytope K having all its vertex stars finite-dimensional and a continuous f: $Y \to K$ such that $\{f^{-1}(\text{vertex star})\}$ refines $\{U\}$. [Reviewer's remark: This result is well-known: "if" is trivial from the paracompactness of CW polytopes, "only if" is a standard result in this field, and the finite dimensionality of the vertex stars is immediate from a result of C. H. Dowker [Amer. J. Math. **69**, 200–242 (1947), p. 207; MR **8**, 594].] *J. Dugundji.*

Albrecht, F. Au sujet de l'espace des ensembles fermés. Com. Acad. R. P. Române **2**, 209–212 (1952). (Romanian. Russian and French summaries)

Albrecht, Felix. Sur l'espace des ensembles fermés d'un espace compact. Com. Acad. R. P. Române **3**, 13–17 (1953). (Romanian. Russian and French summaries)

Albrecht, Felix. Deux théorèmes sur l'espace des ensembles fermés. Com. Acad. R. P. Române **3**, 193–196 (1953). (Romanian. Russian and French summaries)

If E is a uniform space with uniform structure $\mathcal{U}$, $\bar{\mathcal{U}}$ will denote the associated uniform structure on the space $\mathcal{F}(E)$ of closed subsets of E. In these papers the author proves: (I) $\bar{\mathcal{U}}_1$ is finer than $\bar{\mathcal{U}}_2$ if and only if for every $X \in \mathcal{F}(E)$ and every $V_2 \in \mathcal{U}_2$ there exists a $V_1 \in \mathcal{U}_1$ such that 1) $V_1(X) \subset V_2(X)$; 2) for each $x \in X$, there exists an $x' \in X$ such that $V_1(x') \subset V_2(x)$. (II) E is compact and connected if and only if $\mathcal{F}(E)$ is also. (III) E is compact and locally connected if and only if $\mathcal{F}(E)$ is also. (IV) If E is the real line with the additive uniform structure, $\mathcal{F}(E)$ is not connected. (V) If (E_α) is a family of compact spaces, then $\prod \mathcal{F}(E_\alpha)$ is a retract of $\mathcal{F}(\prod E_\alpha)$. *M. E. Shanks* (West Lafayette, Ind.).

Sharp, Henry, Jr. Strongly topological imbedding of F_σ-subsets of E_n. Amer. J. Math. **75**, 557–564 (1953).

Let E_n be Euclidean n-space. Let M_n^k be the set of all points of E_n having at most k rational coordinates. Let $P(n, k, j)$ $(j \leq k)$ be the proposition that for every j-dimensional F_σ-set $S_n^j \subset E_n$ there is a homeomorphism h, of E_n onto itself, such that $h(S_n^j) \subset M_n^k$. The purpose of this paper is to determine for what values of n, k, and j the proposition $P(n, k, j)$ is true. (Only the case $j \leq k$ needs to be considered, because dim $M_n^k=k$.) It is known that $P(n, n, j)$ and $P(n, n-1, j)$ always hold. It was shown by S. W. Hahn [Trans. Amer. Math. Soc. **70**, 301–311 (1951); MR **12**, 728] that $P(2, 0, 0)$ and $P(3, 1, 0)$ are true, and that $P(3, 0, 0)$ is false. The author shows that $P(n, k, j)$ is true for $n \geq 4$, $k=n-2$; he shows further that if S_n^j is a j-dimensional F_σ-set in E_n, with $j \leq n-2$, and H is the space of all homeomorphisms of E_n onto itself, throwing S_n^j into M_n^{n-2}, then H is not only non-empty, but forms a dense G_σ subset of the space of all homeomorphisms of E_n onto itself. It is shown next using an example due to Blankinship [Ann. of Math. (2) **53**, 276–297 (1951); MR **12**, 730], that for $n \leq 4$ and $k<n-2$, $P(n, k, j)$ is always false, even for $j=0$. Thus it remains only to consider $P(3, 1, 1)$. Using an example due to the referee of a paper of Frankl and Pontrjagin [Math. Ann. **102**, 785–789 (1930)], the author shows that $P(3, 1, 1)$ is false. *E. E. Moise* (Ann Arbor, Mich.).

Hamstrom, Mary-Elizabeth. Concerning the imbedding of upper semicontinuous collections of continua in continuous collections of continua. Amer. J. Math. **76**, 793–810 (1954).

Throughout this paper, space is the plane. Let G and W be collections of disjoint sets. If (1) G is a refinement of W, and (2) every element of W contains exactly one

element of G, then G is imbedded in W. The author is concerned with uppersemicontinuous collections G of disjoint, locally connected, atriodic continua filling up a (compact) continuum M in the plane, such that G forms a topological arc, under the natural topology, with the elements of G considered as points. (A locally connected continuum is atriodic if it contains no set homeomorphic to the letter Y.) G^* denotes the union of the elements of G. The end-elements of G are denoted by a and b. An arc $h \in G$ is a K-curve of G if it is true that there are simple closed curves J_1, J_2, with disjoint interiors I_1, I_2, such that (1) $J_1 \cap J_2$ is an arc which contains h and is separated by h, (2) $J_1 \cap J_2 \cap G^* = h$, (3) I_1 does not intersect any element of the interval ah of G, and (4) I_2 does not intersect any element of the interval hb of G. A simple closed curve $J \in G$ is a K-curve of G if (1) K separates two points of G^* in the plane, or (2) J is an end-element of G. A degenerate element $p \in G$ is a K-curve of G if p is not the sequential limiting set of any sequence of simple closed curves of G. Let n be a positive integer, and let H be a closed subcollection of arcs of G. Then $H^{(n)}$ is the set of all elements g of H such that if (1) ε is a positive number and (2) U is an open interval of G, containing g, then there is an arc h, lying in an element g' of $U \cap H$, such that (i) the distance between the end-points of h is less than ε, but (ii) the diameter of h is not less than $1/n$. Theorem I. Suppose that (1) the end-elements of G are simple closed curves, one of which is interior to the other, (2) every non-end-element of G is acyclic, (3) if H is a closed subcollection of arcs of G, then for no natural number n is it true that $H^{(n)} = H$, and (4) each element of G is a K-curve of G. Then there is a continuous collection W of simple closed curves, such that G is imbedded in W. Theorem II. For G to be imbeddable in a continuous collection of simple closed curves, it is necessary and sufficient that (i) G satisfies (3) and (4) of Theorem I, and (ii) given any two simple closed curves belonging to G, one of them is in the interior of the other. Analogous imbedding theorems are proved, in which the containing continuous collection W appears as (1) a collection of arcs and simple closed curves or (2) a collection of continua each of which is homeomorphic to the unique element of G which it contains. The proofs are long.

E. E. Moise (Ann Arbor, Mich.).

Dyer, Eldon. Continuous collections of decomposable continua on a spherical surface. Proc. Amer. Math. Soc. **6**, 351–360 (1955).

R. D. Anderson has given a characterization of the spaces $\mathfrak{G}$ which are the hyperspaces of continuous collections G of (disjoint) nondegenerate locally connected continua, filling up a plane continuum M [G fills up M if M is the union G^* of the elements of G, same Proc. **3**, 647–657 (1952); MR **14**, 783]. The present paper investigates the structure of $\mathfrak{G}$ for the case in which G is a continuous collection of decomposable continua filling up a continuum in the 2-sphere S. The basic lemma is as follows: Theorem I. If G^* is a subset of the boundary of a connected open set lying in $S - G^*$, then there is a subcollection R of G which forms a dense inner limiting set (= dense G_δ-set) in $\mathfrak{G}$, such that each element of R is a local separating point of $\mathfrak{G}$. From this, the following results are proved. Theorem II. Let $ab \in G$; let $\mathfrak{H}$ be an irreducible subcontinuum of $\mathfrak{G}$ from a to b; let α be an arc in $(S - H^*) \cup a \cup b$ which is irreducible from a to b; and let U be a component of $S - (H^* \cup \alpha)$, containing α in its boundary. Then H^* is not a subset of the boundary of U. Theorem III. $\mathfrak{G}$ is a continuous curve (=locally connected continuum). Theorem IV. Every subarc of $\mathfrak{G}$ contains uncountably many local separating points of $\mathfrak{G}$. Theorem V. Every nondegenerate subcontinuum of $\mathfrak{G}$ contains uncountably many local separating points of $\mathfrak{G}$. Theorem VI. If G fills up S, then $\mathfrak{G}$ is a dendron. Theorem VII. If G fills up S, and no element g of G is the boundary of three different components of $S - g$, then $\mathfrak{G}$ is an arc. Theorem VIII. For each dendron D, there is a continuous collection G of decomposable continua filling up S, such that $\mathfrak{G}$ is homeomorphic to D. (The author remarks that Theorem VIII is a corollary of an unpublished result of Anderson, and then gives an indication of a direct proof.)

E. E. Moise (Ann Arbor, Mich.).

Bing, R. H. Partially continuous decompositions. Proc. Amer. Math. Soc. **6**, 124–133 (1955).

Let G be a collection of disjoint compact sets, in a metric space S. Suppose that for each $g \in G$, and each $\varepsilon > 0$, there is a $\delta > 0$ such that if $g' \in G$, and $\varrho(g, g') < \delta$, then $g \subset V(g', \varepsilon)$. Then G is lower-semicontinuous. (Here $\varrho(g, g')$ is the distance between the compact sets g, g'; and $V(g', \varepsilon)$ is the ε-neighborhood of g' in S.) Let X be the union of the elements of G. G will be called a decomposition of X. If the elements of G are connected, then G is called monotone. By a continuum the author means a closed connected set (not necessarily compact.) Theorem I. Let G be a monotone lower-semicontinuous decomposition of a locally compact continuum X, and suppose that G has more than one element. Let $g_0 \in G$, and let ε be a positive number. Then there is a compact continuum M such that (1) M is a union of elements of G, (2) M properly contains g_0, and (3) if g, $g' \in G$, and g, $g' \subset M$, then the Hausdorff distance between g and g' is less than ε. Theorem II. Let G be a lower-semicontinuous decomposition of a locally compact separable metric space X; and suppose that G is uncountable. Then there is an uncountable subcollection G' of G, such that G' is continuous and such that the union of the elements of G' is compact. Theorem III. Let G be a monotone lower-semicontinuous decomposition of a compact connected metric space X. Let A and B be disjoint closed non-empty subsets of X. Then there is a continuum M such that (1) M intersects both A and B, (2) M is a union of elements of G, and (3) M is irreducible with respect to the property of being a continuum in X, satisfying (1) and (2). Moreover, if M_A and M_B denote the union of all elements of G that lie in M and intersect A and B respectively, then each of the sets $M - M_A$, $M - M_B$, and $M - (M_A \cup M_B)$ is connected. Theorem IV. Let G be a lower-semicontinuous decomposition of a closed plane set, such that each element of G is a continuous curve (=locally connected compact continuum) containing more than one point. Then G is also upper-semicontinuous, and hence is continuous. Various examples are given, indicating directions in which the above results cannot be extended. It is explained that lower-semicontinuity and upper-semicontinuity are related to certain properties of mappings in the following way: Let f be a function (not necessarily continuous) of a compact metric space X onto a topological space $Y = f(X)$, such that each set $f^{-1}(y)$ $(y \in Y)$ is closed. Let G be the collection consisting of all inverse images $f^{-1}(y)$ of points y of Y. Various relations are given between semicontinuity properties of G and topological properties of f and Y; for upper-semicontinuity, these relations are of course well-known.

E. E. Moise (Ann Arbor, Mich.).

Gordon, I. I. Generalization of Kakutani's theorem on continuous functions given on a sphere. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 97–99 (1955). (Russian)

Let a continuous real-valued function f be given on the surface of a sphere S^2. Let Δ_0 be a given triplet of points forming an isosceles triangle on S^2. Theorem: There exists on S^2 a triplet of points Δ, congruent to Δ_0, such that the values of f at the three points Δ are equal. This generalizes the Kakutani theorem [S. Kakutani, Ann. of Math. (2) **43**, 739–741 (1942); MR **4**, 111] in which Δ_0 was on orthogonal triad; the case of a general scalene triangle remains open. The proof is only a trivial variation of that given by H. Yamabe and Z. Yujobô [Osaka Math. J. **2**, 19–22 (1950); MR **12**, 198] for their n-dimensional version of Kakutani's theorem. *F. J. Dyson.*

Sikorski, R. On the operation of cutting off mappings. Prace Mat. **1**, 136–140 (1955). (Polish. Russian and English summaries)

Let X be a metric space, Y a bounded metric space and Y^X the space of all continuous functions $f: X \to Y$ with the usual distance. If $F \subset X$ then $\Phi(f): F \to Y$ denotes the restriction $f|F$ of f. The author proves that if F is closed and Y is an absolute neighborhood retract then the map $\Phi: Y^X \to Y^F$ is open (i.e., Φ maps open sets onto open sets). *S. Eilenberg* (New York, N.Y.).

Ganea, Tudor. Partitions régulières non normales des espaces normaux. Com. Acad. R. P. Romîne **5**, 37–41 (1955). (Romanian. Russian and French summaries)

It is shown that the Prüfer manifold is a non-normal space which is the quotient of a metrisable space with respect to an open equivalence relation. *M. Heins.*

Ganea, Tud. Zur Unikohärenz cartesischer Produkte. Com. Acad. R. P. Române **1**, 315–317 (1951). (Romanian. Russian and German summaries)

The theorem proved here is proved again in another of the author's papers [Ann. Soc. Polon. Math. **25**, 30–42 (1953); MR **15**, 146]. *R. H. Fox* (Princeton, N.J.).

Ganea, T. Transformations à petites tranches. Com. Acad. R. P. Române **1**, 41–43 (1951). (Romanian. Russian and French summaries)

The results of this paper are all contained in another of the author's papers [Fund. Math. **38**, 179–203 (1951); MR **14**, 193]. *R. H. Fox* (Princeton, N.J.).

Ganea, T. Opérations à ensembles simplement connexes. Com. Acad. R. P. Române **1**, 147–149 (1951). (Romanian. Russian and French summaries)

The theorems announced are proved in another of the author's papers [Fund. Math. **38**, 179–203 (1951); MR **14**, 193]. *R. H. Fox* (Princeton, N.J.).

Pavel, Monica. Deux théorèmes sur les rétractes. Com. Acad. R. P. Române **2**, 189–191 (1952). (Romanian. Russian and French summaries)

If two absolute neighborhood retracts are quasi-homeomorphic, and one is an absolute retract, then so is the other one. If, for every position ε, an absolute neighborhood retract has an absolute retract as an ε-image, then it is also an absolute retract. If the intersection of a decreasing sequence of absolute retracts is an absolute neighborhood retract, it is an absolute retract.

R. H. Fox (Princeton, N.J.).

★**Errera, A. Sur un théorème de M. Whitney, un problème de Lebesgue et les réseaux de Tait.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 51–55. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author considers the possibility of classifying cubic maps on the sphere by means of the two trees determined in such a map by a Hamiltonian circuit in the dual map. He discusses the 4-colour reducibility of cubic maps in terms of P. G. Tait's 3-colourings of the edges.

W. T. Tutte (Toronto, Ont.).

Algebraic Topology

★**Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1948/1949. Topologie algébrique. 2ème éd.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. iii+86 pp. (polycopiées).

New edition of the first eleven exposés of the first edition [MR **14**, 670].

Aleksandrov, P. On the homeomorphism of point sets. Dokl. Akad. Nauk SSSR (N.S.) **97**, 757–760 (1954). (Russian)

This is a brief description of the results derived in the paper reviewed below. *H. Samelson* (Ann Arbor, Mich.).

Aleksandrov, P. S. On homeomorphism of point sets. Trudy Moskov. Mat. Obšč. **4**, 405–420 (1955). (Russian)

An abstract spectrum consists of an inverse system of star-finite simplicial complexes, where the maps π_α^β involved associate with any vertex of the complex β the set of vertices of some simplex of α ($\beta > \alpha$), and where the composition property $\pi_\alpha^\beta \cdot \pi_\beta^\gamma e \subset \pi_\alpha^\gamma e$ holds for the vertices of the complex γ ($\gamma > \beta > \alpha$). A cofinal part is defined in the natural fashion (the projections π'^β_α of a cofinal part can differ from the π_α^β but satisfy $\pi'^\beta_\alpha e \subset \pi_\alpha^\beta e$). The triangulations (locally finite simplicial polyhedra) containing a given set E in Euclidean space R^n form the geometrical spectrum of E in R^n, under appropriate definition of ordering and of the maps π_α^β (a vertex of β goes into the set of vertices of its carrier in α). Main theorem: Two subsets E, resp. E', of R^n, resp. $R^{n'}$, are homeomorphic if and only if there exists an abstract spectrum, which has two cofinal parts s, resp. s', which also are cofinal parts of the geometrical spectrum of E, resp. E'. The proof rests on the consideration of the space associated with an abstract spectrum (constructed on the model of inverse limit spaces) and of several related types of spectra, associated with a set E in n-space. *H. Samelson.*

Aleksandrov, P. S. On some consequence of Sitnikov's second duality law. Dokl. Akad. Nauk SSSR (N.S.) **96**, 885–887 (1954). (Russian)

Several questions, raised by the author [Mat. Sb. N.S. **21** (63), 161–232 (1947); MR **9**, 456] can be answered using Sitnikov's second duality law [see the paper reviewed below]. Kernel and image of the natural homomorphism h of $\Delta_s{}^p A$ into $\delta^p A$ (A, B are complementary subsets of the sphere S^n) are dualized, i.e., invariants of B; in fact, every property of A that can be expressed in terms of h is dualized. The proof rests on the existence of a dual homomorphism $\bar{h}$ of $\bar{\Delta}^q B$ into $\bar{\delta}^q B$, with properties

dual to h. Some related results and some open questions, e.g. the existence of a duality by linking, equivalent to Sitnikov's first duality law, are discussed briefly.

H. Samelson (Ann Arbor, Mich.).

Sitnikov, K. New duality relations for nonclosed sets. Dokl. Akad. Nauk SSSR (N.S.) **96**, 925–928 (1954). (Russian)

Brief description of a proof of a conjecture of P. S. Aleksandrov (1935), namely that the homology groups with compact carriers of an arbitrary subset A of spherical space S^n are "dualized", i.e., are invariants of the complement B of A. Specifically, in the Sitnikov cohomology group $\nabla^p B$ a subgroup $N^p_\nabla B$ is singled out (annihilator of the p-cycles of B over the dual coefficient group), and $\Delta^p_c A \approx \nabla^q B - N^q_\nabla B$ $(p+q=n-1)$; this is the second general duality law; instead of S^n one can take any homology n-manifold, acyclic in dimensions p, $p+1$. This is contained in the duality isomorphism theorem: There is a natural isomorphism between $\nabla^p A$ and $\Delta^q B$, which sends $N^p_{\nabla_c} A$ into $N^q_{\Delta_c} B$, and $N^p_\nabla A$ into $H^q B$; in the case of compact coefficients $H^q B$ is replaced by $N^q_\Delta B$. The subgroups occurring here have invariant definitions similar to that of N^p_∇; they are all "dualized," and this holds also for the projective groups $\delta^p A$ and their modifications $\overline{\delta^p} A$. Finally, an intrinsic characterization is given of the quasi-closed sets [cf. Sitnikov, Mat. Sb. N.S. **34(76)**, 3–54 (1954); MR **16**, 736]; and the maximal duality domain in the sense of Aleksandrov (reference in the preceding review) is shown to consist of the sets which are simultaneously quasi-closed and quasi-open. *H. Samelson.*

Chang, S. C. On the homotopy groups of $S^p \cup S^q$. I, II. Acta Math. Sinica **3**, 186–189, 190–199 (1953). (Chinese. English summary)

An English version has appeared in Sci. Sinica **3**, 225–236 (1954); MR **16**, 847.

Pavel, Monica. Les classes d'homotopie des transformations également continues. Com. Acad. R. P. Române **3**, 115–116 (1953). (Romanian. Russian and French summaries)

Let X and Y be compact metric spaces and let F be any set of continuous maps of X into Y. The author proves the following theorem: If the maps in F are equi-continuous and if Y is an ANR, then the maps in F are contained in a finite number of homotopy classes. *S. T. Hu.*

Pavel, Monica. Une propriété locale des espaces fibrés. Com. Acad. R. P. Române **2**, 501–503 (1952). (Romanian. Russian and French summaries)

Let X be a fiber space over B relative to a projection $p: X \to B$ in the reviewer's sense [Proc. Amer. Math. Sc. **1**, 756–762 (1950); MR **12**, 435]. The author proves that if X is an ANR, then so is B. *S. T. Hu* (Athens, Ga.).

Milnor, John. Link groups. Ann. of Math. (2) **59**, 177–195 (1954).

An ordered collection $(l_1, \cdots, l_n)$ of homeomorphisms of the circle C into an open 3-dimensional manifold M is called a (proper) n-link L if the images $l_1(C), \cdots, l_n(C)$ are pairwise disjoint. Two links L and L' are called homotopic if there exist homotopies h_i between the maps l_i and the maps l_i' such that, for each value of the time parameter t, the sets $h_1(C, t), \cdots, h_n(C, t)$ are pairwise disjoint. During such a deformation, then, each component of the link is allowed to cross itself, but no two components are ever allowed to intersect. Clearly the "group $G(L)$ of the link L", i.e. the fundamental group of the complement in M of $|L| = \mathbf{U}_{i=1}^n l_i(C)$, is not preserved under "homotopy". On the other hand, it is well known that the 1-dimensional homology group $H(L)$ has the following properties: (1) it is unaltered by homotopy of L, and (2) it is a homomorphic image of $G(L)$. Of all the groups that have these two properties in common with $H(L)$, the link group defined by the author is in some sense the "largest" that one could possibly define.

To be precise, the link group $\mathcal{G}(L)$ of a link L is defined to be $G(L)/E(L)$ where $E(L)$ is defined as follows: Let $L^i = (l_1, \cdots, \hat{l}_i, \cdots, l_n)$ denote the $(n-1)$-sublink obtained by deleting the ith component, and let $A_i(L)$ denote the kernel of the injection homomorphism $G(L) \to G(L^i)$. Then $E(L) = [A_1][A_2] \cdots [A_n]$, where $[A]$ denotes the commutator subgroup of A. Crudely expressed, $\mathcal{G}(L)$ is obtained from $G(L)$ by allowing to commute any two elements which are prevented from being trivial by, say, only the ith component. The link group of an n-link is nilpotent of class n.

In the link group $\mathcal{G}$ there is a "peripheral" structure [cf. R. H. Fox, Nederl. Akad. Wetensch. Proc. Ser. A. **55**, 37–40 (1952); MR **13**, 966] consisting of a meridian α_i, an element of $\mathcal{G}$, and a parallel $\beta_i \mathcal{A}_i$, a coset of the kernel $\mathcal{A}_i$ of the injection homomorphism $\mathcal{G}(L) \to \mathcal{G}(L^i)$. The pair $(\alpha_i, \beta_i \mathcal{A}_i)$ is determined only up to inner automorphisms of $\mathcal{G}$. It is proved that, if two links are homotopic, their link groups are isomorphic, and in such a way that each ith meridian-parallel pair of the one group corresponds to an ith meridian-parallel pair of the other.

A link L is trivial if it is homotopic to some L' where each component $l_i'(C)$ is contained in a 3-cell and these cells are pairwise disjoint. The author considers a link L^* obtained from a link L by replacing each component by a collection of parallels whose mutual linking numbers are zero. If $n=1$ the link L^* is always trivial; if $n \geq 1$ this need not be true.

A link L is i-trivial if it is homotopic to some L' whose ith component is contained in a 3-cell whose complement contains all the other components. If f and g are homeomorphic maps of C into the complement of $|L|$, they represent the same element of $\mathcal{G}(L)$ if and only if the $(n+1)$-link $(L, g \cdot f^{-1})$ is $(n+1)$-trivial; they represent conjugate elements of $\mathcal{G}(L)$ if and only if the $(n+1)$-links (L, f) and (L, g) are homotopic.

Denote by K_i and $\mathcal{K}_i$ the kernels of the injection homomorphisms $JG(L) \to JG(L_i)$ and $J\mathcal{G}(L) \to J\mathcal{G}(L^i)$ of the integral group rings $JG(L)$ and $J\mathcal{G}(L)$. The rings $JG/\Sigma_{i=1}^n K_i^2$ and $J\mathcal{G}/\Sigma_{i=1}^n \mathcal{K}_i^2$ are isomorphic, and are denoted by $\mathcal{R} = \mathcal{R}(L)$. If the link L is i-trivial, each element of $\mathcal{A}_i(L)$ is expressed uniquely in the form α_i^σ where $\sigma \in \mathcal{R}(L^i)$. If L is a trivial n-link, each element of $\mathcal{G}(L)$ can be expressed uniquely as the product of an element of $\mathcal{G}(L^n)$ and an element of $\mathcal{A}(L)$. Hence it follows by induction that each element of $\mathcal{G}(L)$ can be expressed uniquely in the form $\varphi \alpha_1^{\sigma_1} \alpha_2^{\sigma_2} \cdots \alpha_n^{\sigma_{n-1}}$, where

$$\varphi \in F = \pi(M), \quad \sigma_0 \in JF, \quad \sigma_1 \in \mathcal{R}(l_1), \quad \sigma_2 \in \mathcal{R}(l_1, l_2), \cdots, \quad \sigma_{n-1} \in \mathcal{R}(l_1, l_2, \cdots, l_{n-1}).$$

If the word-problem can be solved in F it is shown how to solve the word-problem in the rings $\mathcal{R}(l_1, \cdots, l_r)$, $0 \leq r \leq n-1$, and thus how to solve the word problem in the link group $\mathcal{G}(L)$ of a trivial link L. Using this, the following is an inductive procedure for deciding whether an arbitrary given n-link $(l_1, \cdots, l_n)$ is trivial: l_1 is trivial if and only if it represents the trivial element of F; if

$(l_1, \cdots, l_{r-1})$ has been found to be trivial then $(l_1, \cdots, l_r)$ is trivial if and only if l_r represents the trivial element of $\mathcal{G}(l_1, \cdots, l_{r-1})$. It is shown that if the link group of an n-link L is isomorphic to the link group of some trivial n-link in an isomorphism that preserves the conjugate classes corresponding to meridians, then L is trivial.

Turning to the case M=euclidean space, it is shown that $\mathcal{G}(L)$ then has a presentation of an especially simple form:

$$\mathcal{G}(L)=(\alpha_1, \cdots, \alpha_n : \alpha_1 w_1 = w_1 \alpha_1, \cdots, \alpha_n w_n = w_n \alpha_n, E=1),$$

where α_i is a meridian to the ith component, $w_i = w_i(\alpha_1, \cdots, \alpha_n)$ is a word representing the ith parallel $\beta_i \in \mathcal{A}_i$, and E denotes a set of relations that specify that each α_i commutes with its conjugates. An n-link in euclidean space is called almost trivial if every proper sublink is trivial. If L is almost trivial the element β_s' of $\mathcal{G}(L^s)$, determined by the parallel $\beta_s \in \mathcal{A}_s$, can be expressed in the form α_r^σ, and the element $\sigma \in \mathcal{R}(L^{r,s})$ can be written uniquely in the form

$$\sigma = \sum \mu(i_1, \cdots, i_{n-2}, r, s)(\alpha_{i_1} - 1) \cdots (\alpha_{i_{n-2}} - 1),$$

where the summation extends over all permutations of the $n-2$ integers $1, 2, \cdots, \hat{r}, \cdots, \hat{s}, \cdots, n$. The $n!$ integers $\mu(i_1, \cdots, i_n)$ are homotopy invariants of L. These integers are not independent, but satisfy the following (complete) set of identities:

$$\mu(i_1, i_2, \cdots, i_{n-1}, i_n) = \mu(i_n, i_1, i_2, \cdots, i_{n-2}, i_{n-1}),$$

$$\mu(i_1, \cdots, i_\nu, r, j_1, \cdots, j_{n-\nu-2}, s) = (-1)^{n-\nu} \sum \mu(h_1, \cdots, h_{n-2}, r, s),$$

where the summation is extended over all sets $h_1, \cdots, h_{n-2}$ of integers formed by intermeshing $i_1, \cdots, i_\nu$ with $j_{n-\nu-2}, \cdots, j_2, j_1$ in that order. If the orientation of one component of L is reversed, each μ is multiplied by -1; if the orientation of space is reversed, each μ is multiplied by $(-1)^{n-1}$.

Every link (in euclidean space) with two components is almost trivial. The linking number is $\mu(1, 2)$, and is thus a complete invariant. A link with three components is almost trivial if and only if each of the three linking numbers is zero; a complete set of invariants is the single integer $\mu(1, 2, 3)$. In euclidean space an arbitrary link with three components is completely specified up to homotopy by the three linking numbers $\mu(1, 2)$, $\mu(1, 3)$, $\mu(2, 3)$ and the residue class of $\mu(1, 2, 3)$ modulo $(\mu(1, 2), \mu(1, 3), \mu(2, 3))$. Thus the number of homotopically distinct 3-links with prescribed linking numbers $\mu(1, 2)$, $\mu(1, 3)$, $\mu(2, 3)$ is just the greatest common divisor of $\mu(1, 2)$, $\mu(1, 3)$, $\mu(2, 3)$. The author indicates how these explicit considerations may be extended to n-links with $n>3$.

If, in an arbitrary open 3-dimensional manifold M, one considers all the possible link groups, together with their peripheral structure, and their injections into $\pi(M)$, one has what the author calls the link structure of M. This is a topological invariant of M, but it is not known whether it is also an invariant of the homotopy type of M. Several years ago the author and the reviewer tried to decide this question for the lens spaces, but were unable to reach a decision. If the link structure should turn out to be a homotopy-type invariant it would lead to an intensive generalization of the duality theorems of homology groups.

R. H. Fox (Princeton, N.J.).

Al'ber, S. I. Homologies of homogeneous spaces. Dokl. Akad. Nauk SSSR (N.S.) **98**, 325–328 (1954). (Russian)

Extending previous work [same Dokl. (N.S.) **91**, 1237–1240 (1953); MR **15**, 457], the author indicates a recursive method to determine quickly the homology of various homogeneous spaces; as examples he treats the Grassman manifolds $G(n, k)$ (great k-spheres on the n-sphere) and $C_0(n, k)$ (all k-spheres on the n-sphere with all point-spheres identified). $G(n, k)$ contains $G(n-1, k)$ and $G(n-1, k-1)$ in a natural way, and there is a deformation retraction of the complement of $G(n-1, k-1)$ onto $G(n-1, k)$. Using this deformation one obtains a series of theorems describing the homology (integral and mod 2) of $G(n, k)$ and the homology type of the inclusion $G(n-1, k) \subset G(n, k)$ (always an isomorphism into except in the case: integral coefficients, n odd, k even). $C_0(n, k)$ is treated similarly.

H. Samelson (Ann Arbor, Mich.).

GEOMETRY

Lauffer, R. Zur Geometrie eines Euklidischen Dreiecks. Simon Stevin **30**, 176–177 (1955).

★ **Zetel', S. I. On a theorem of the geometry of a triangle and its projective generalization.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 56–58. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Let P_1, P_2, P_3 be three points on the respective sides of a triangle $A_1A_2A_3$ with A_1P_1, A_2P_2, A_3P_3 concurrent at a point P and let Q_j $(j=1, 2, 3)$ be the fourth harmonic point to P_j with respect to the endpoints of the side of the triangle on which P_j is located. The midpoints of the three line segments P_jQ_j $(j=1, 2, 3)$ are then collinear. The author proves this theorem and discusses several particular cases and also generalizations. *E. Lukacs.*

Kárteszi, Ferenc. Some planimetric relations obtained through stereometric way. Mat. Lapok **3**, 47–58 (1952). (Hungarian. Russian and English summaries)

Hanke, W. Affine Abbildung. Optik **12**, 322–328 (1955).

The author modifies a procedure suggested by K. Schwidefsky [Optik, **2**, 434–444 (1947); MR **9**, 458] which produces an affine transformation by means of a restitutional projector. The author's procedure is more complicated but has a wider range of applicability than Schwidefsky's. *E. Lukacs* (Washington, D.C.).

Hanke, W. Optische Umwandlung des allgemeinen Vierecks zum Parallelogramm. Optik **12**, 316–321 (1955).

The author discusses the projection of a plane quadrilateral into a parallelogram. This transformation has applications in cartography.

E. Lukacs (Washington, D.C.).

Krames, Josef. Elementargeometrischer Nachweis des „gefährlichen" Drehzylinders beim räumlichen Rückwärtseinschnitt. Elem. Math. **10**, 106–108 (1955).

Varga, Tamás. Der Satz vom W. Bolyai über die Verlegungsgleichheit von Polygonen. Mat. Lapok **5**, 101–114 (1954). (Hungarian. Russian and German summaries)

The following noteworthy result is due to the Hungarian mathematician Bolyai Farkas, the father of Bolyai János: If two polygons A and B have the same area, then A can be divided into a finite number of polygons $A_1, A_2, \cdots, A_n$ and B can be divided into a finite number of polygons $B_1, B_2, \cdots, B_n$, $n \geq 1$, in such a way that A_i is congruent with B_i for $1 \leq i \leq n$. This interesting theorem is here brought before the public in its country of origin, and three proofs of it are given: one essentially due to Bolyai Farkas himself [Wolfgangi Bolyai de Bolyai, Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris, methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici, t. II, Marosvásárhely, 1832–1833; 2nd ed., Acad. Sci. Hungar., Budapest, 1904, pp. 108–109 (in Latin and in German he called himself Wolfgang Bolyai)]; a simplification of this proof due to the author of the present paper, and a proof due to V. F. Kagan [On transformation of polyhedra, 2nd ed., Gostehizdat, 1933; see also B. V. Kutuzov, Geometriya, Učpedgiz, 1950, pp. 209–212]. *G. A. Dirac.*

Fejes Tóth, L. Extremum properties of the regular polytopes. Acta Math. Acad. Sci. Hungar. **6**, 143–146 (1955). (Russian summary)

Using Steiner's process of symmetrization, the author proves that, among all convex four-dimensional polytopes of given in-sphere, bounded by 5, 16 or 600 tetrahedra, the corresponding regular polytope $\{3, 3, r\}$ ($r=3$, 4, or 5) has the minimal hyper-surface (volume), and therefore the minimal content. He also proves the analogous result for the polytope $\{3, 4, 3\}$ whose cells are 24 octahedra, and he conjectures it for the remaining two regular polytopes. Finally, he proves that the ratio of circum-radius to in-radius is at least n for an n-dimensional simplex, and at least $\sqrt{n}$ for an n-dimensional polytope isomorphic with the measure polytope (cube). *H. S. M. Coxeter.*

Fischer, Helmut Joachim. Geometrische Netze und Konfigurationen und ihre Beziehungen zur Vektorrechnung und Zahlentheorie. III. Collect. Math. **7**, 3–67 (1954).

The author carries out the three-dimensional analogue (and briefly describes the n-dimensional analogue) of his two-dimensional investigation in part I [Collect. Math. **4**, no. 2, 57–119 (1951); MR **14**, 785]. Five points $P_1, \cdots, P_5$, no four coplanar, can be joined in threes by ten planes, which have ten further points of intersection; these can be joined to the original points and to one another by further planes, which intersect in further points, and so on. The system of points and planes arising in this manner is called a "net" [Möbius, Der barycentrische Calcül, Barth, Leipzig, 1827, §§ 206–210]. Any five independent points of the net can be used instead of the P's for constructing the whole net [ibid., § 212]. If $P_1P_2P_3P_4$ is the tetrahedron of reference, while P_5 is (1, 1, 1, 1), the net consists of all the points and planes whose coordinates are integers [ibid., § 211]. The author emphasizes the self-duality of the net by showing that it can be derived just as well from five planes $E_1, \cdots, E_5$, whose lines of intersection $E_i \cdot E_m$ lie in the planes $P_iP_jP_k$ while their points of intersection $E_i \cdot E_j \cdot E_k$ lie on the lines P_iP_m. The five P's and five E's are said to be of degree 0. Any other element of the net is said to be of degree n if it is the join or intersection of three elements, one of degree $n-1$ and the others the same or lower. Thus he finds one family of degree 1, four of degree 2, 173 of degree 3, and he estimates that there must be thousands of millions of degree 4. He also describes some of the configurations formed by points and planes of degree 3 or less. *H. S. M. Coxeter.*

Narain, Shanti, and Ram Behari. A note on classification of quadrics by the matrix method. Bull. Calcutta Math. Soc. **46**, 163–167 (1954).

Mihăileanu, N. Une interprétation de la géométrie de Lobatchevsky sur l'hyperboloïde à deux nappes. Com. Acad. R. P. Romîne **4**, 579–580 (1954). (Romanian. Russian and French summaries)

Edge, W. L. 31-point geometry. Math. Gaz. **39**, 113–121 (1955).

This is a detailed treatment of various aspects of the projective plane with 31 points. Special properties of cross-ratios, conics, collineations, and polygons are all discussed. *Marshall Hall, Jr.* (Columbus, Ohio).

Segre, Beniamino. Ovals in a finite projective plane. Canad. J. Math. **7**, 414–416 (1955).

In a finite projective plane with $n+1$ points on a line there can be at most $n+2$ points with the property that no three are on a line, and if n is odd there can be at most $n+1$ with this property. If n is even and we have $n+1$ points, no three on a line, then there exists a further point which can be adjoined to these giving $n+2$ points, no three on a line. In a Desarguesian plane a non-degenerate conic contains $n+1$ points, no three on a line. If, when n is odd, we call $n+1$ points, no three on a line, an oval, then it was conjectured by Järnefelt and Kustaanheimo [Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, Tanum, 1952, pp. 166–182; MR **14**, 1008] that in a Desarguesian plane of odd order n, an oval is necessarily a conic. This conjecture is shown to be true in this paper. The method of proof is ingenious. We may take three points of the oval to be A_1: (1, 0, 0), A_2: (0, 1, 0), and A_3: (0, 0, 1) and if $P(a_1, a_2, a_3)$ is a further point on the oval and $x_3=\lambda_1 x_2$, $x_2=\lambda_2 x_1$, $x_1=\lambda_3 x_3$ are the three secants PA_1, PA_2, PA_3, then immediately $\lambda_1\lambda_2\lambda_3=1$. Since the product of all non-zero elements in the field is -1, it will follow that for the tangents at A_1, A_2, A_3 that $x_2=k_1x_3$, $x_3=k_2x_1$, $x_1=k_3x_2$ we will have $k_1k_2k_3=-1$. From this the inscribed triangle and its circumscribed triangle are perspective with respect to the center $(1, k_1k_2, -k_3)$. It follows generally that every inscribed triangle and its circumscribed triangle are perspective. Using this relation on the triangles formed from P, A_1, A_2, and A_3, we find that the coordinates of P satisfy a quadratic equation which becomes $x_2x_3+x_3x_1+x_1x_2=0$ if we take C as (1, 1, 1), as we may. [The fact that this conjecture seemed implausible to the reviewer seems to have been at least a partial incentive to the author to undertake this work. It would be very gratifying if further expressions of doubt were as fruitful.]

Marshall Hall, Jr. (Columbus, Ohio).

Segre, Beniamino. Sulle ovali nei piani lineari finiti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 141–142 (1954).

Summary of the paper reviewed above.

Marshall Hall, Jr. (Columbus, Ohio).

★**Hall, Marshall, Jr. Projective planes and related topics.** California Institute of Technology, 1954. vi+77 pp.

The booklet contains five lectures delivered in April 1954 at the California Institute of Technology. These summarize recent developments in the field of projective planes up until about 1953; each is followed by an adequate bibliography. To quote from the preface: "In recent years there has been a revival in the Foundations of Projective Geometry and there has been a great deal of interest in this subject mainly in this country and Russia." It may be added that, since 1953, an active group of researchers on projective planes has appeared in Germany.

Lecture I, Projective planes, deals with the axioms of incidence, the concept of a free plane, the theorem that every subplane of a free plane is free and the coordinatization of a plane by a planar ternary ring.

Lecture II, Topics in lattice theory, considers projective spaces as lattices. If a projective space is identified with the lattice of its subspaces, the projective spaces are the only lattice-indecomposable atomic relatively complemented modular lattices. A projective plane can be imbedded in a higher-dimensional projective space if and only if the plane is Desarguesian; the author discusses the more complicated question of the imbedding of modular lattices in complemented modular lattices. Turning next to representation of lattices, the author notes that every lattice L is representable as a sublattice $\mathfrak{U}$ of the lattice of all equivalence relations on a suitable set U. Here U and $\mathfrak{U}$ can always be chosen so that, for each pair E_1, E_2 in $\mathfrak{U}$, the union $E_1 \cup E_2$ has one of the (increasingly inclusive) types: (1) $E_1 \cup E_2 = E_1E_2 = E_2E_1$; (2) $E_1 \cup E_2 = E_1E_2E_1 = E_2E_1E_2$; (3) $E_1 \cup E_2 = E_1E_2E_1E_2 = E_2E_1E_2E_1$. Type (2) characterizes the modular lattices. Lattices of type (1) are partially characterized; in particular, among the projective planes, only the Desarguesian ones have representations of type (1).

Lecture III, Topics in the theory of rings, deals with the equivalence of certain (geometric) axioms on a projective plane and (algebraic) axioms on a coordinatizing planar ternary ring. The affine minor axiom of Desargues corresponds to the axioms for a Veblen-Wedderburn system, the (full) Desargues' axiom to the axioms for an associative division ring, the projective minor axiom of Desargues to the axioms for an alternative division ring. A noteworthy feature of the lecture is the complete determination (distilled from ten research papers by eight authors) of the structure of alternative division rings which are not associative. The geometric axioms corresponding to the division rings with the right inverse property are also explored but the fact that these rings are alternative is only partly proved.

Lecture IV is entitled Topics in group theory. One invariant of a projective plane is the group of projectivities of an (arbitrarily chosen) line of the plane into itself. This group is triply transitive. The requirement that the subgroup leaving three distinct points fixed have order one is equivalent to the axiom of Pappus, which implies the axiom of Desargues. A second invariant is the group of collineations. The latter can be very large; in the Desarguesian case it consists of the homogeneous 3-dimensional semi-linear transformations over an associative division ring. And it can consist of the identity only, as witness a suitable infinite plane. If R is a planar ternary ring with ternary operation $x \cdot y \circ z$, the set of permutations $x \to x \cdot m \circ b$ $(m \neq 0)$ is characterized. The set is a group if and only if $x \cdot y \circ z = xy + z$ where $(R, +, \cdot)$ is a near field. A construction is given of essentially all finite near fields. The lecture concludes with a generalization of a theorem of Jordan on quadruply transitive groups.

Lecture V, Topics in matrices and the theory of numbers, imbeds the theory of finite projective planes in that of block designs. A block design D consists of a finite set D of v distinct elements together with b subsets $S_1, \cdots, S_b$ of k elements each such that each element of D lies in r of the S_i and each two distinct elements of D lie in λ of the S_i. Necessarily $bk = vr$, $r(k-1) = \lambda(v-1)$. D is symmetric if $b = v$, $r = k$; in this case each two distinct subsets have λ common elements. A symmetric design with $\lambda = 1$ is a finite projective plane. Each design is characterized by an incidence matrix of dimensions b by v with elements 0 and 1. By matrix methods, in particular, by the theory of rational quadratic forms, a number of non-existence theorems for designs are obtained. Most of these are patterned after the earliest such theorem: a necessary condition for the existence of a finite projective plane with $n+1$ points on each line is, if $n \equiv 1$ or 2 mod 4, that n be a sum of two integral squares. Of a different character is the assumption that the symmetric design D possess a collineation which is transitive and regular on the elements of D; this leads to the theory of difference sets and permits very strong non-existence theorems. The lecture concludes with a method of showing non-existence of certain block designs which are partially given in advance; the method fails in the symmetric case.

R. H. Bruck (Madison, Wis.).

André, Johannes. Projektive Ebenen über Fastkörpern. Math. Z. **62**, 137–160 (1955).

The author continues his study of the affine minor Theorem of Desargues [Math. Z. **60**, 156–186 (1954); MR **16**, 64]. A perspectivity π of a projective plane A is a collineation of A which leaves fixed all the points of a line U (the axis of π) and all the lines through a point c (the center of π). If π is not the identity, U and c are unique. As previously defined by the author, a plane A is a translation plane with absolute line U if the set of perspectivities of A with U as axis and with centers on U is transitive on the points of the affine plane $A-U$. In terms of Hilbert's method of coordinatizing, the translation planes are those with coordinates from a quasifield (Veblen-Wedderburn system with one-sided distributive law in the form $x(y+z) = xy+xz$). Adapting a concept of Baer [Amer. J. Math. **64**, 137–152 (1942); MR **3**, 179], the author calls an ordered pair (p, q) of points of the absolute line U admissible provided that, for each line H through q, other than U, the group of perspectivities with center p, axis H is transitive on the points of $U-\{p\}-\{q\}$. And Z denotes the set of all admissible pairs (p, q).

The translation planes are partitioned into six classes: I. Z is empty. Then A can be coordinatized by a quasifield but not by a distributive quasifield (division ring). II. Z consists of exactly one pair, of form (p, p). Then A can be coordinatized by a distributive quasifield but not by a left alternative division ring. III. Z consists precisely of all pairs (p, p). Then A can be coordinatized by a left alternative division ring but not by a skewfield. [Reviewer's remark. This means that A can be coordinatized by a Cayley division algebra. See, for example, the review of San Soucie, Proc. Amer. Math. Soc. **6**, 291–296 (1955); MR **16**, 896.] IV. Z consists of exactly two pairs (p, q) and (q, p), $p \neq q$. Then A is coordinatized by a nearfield (associative quasifield) which has more than 9 elements and is not a skewfield. Although

A may have coordinate quasifields which are not nearfields, all the coordinate nearfields of A are isomorphic. V. To each p of U there corresponds exactly one q of U such that (p, q) is in Z; moreover, $q \neq p$ and (q, p) is also in Z. Then A is coordinatized by the unique nearfield of 9 elements which is not a field. VI. Z consists of all pairs. Then A is Desarguesian.

The paper concludes with a complete determination of the collineation group of a nearfield plane in terms of the nearfield; the "exceptional" plane (V) gives different results from the planes of class IV.

R. H. Bruck (Madison, Wis.).

Boccioni, Domenico. Spazi affini immersi in uno spazio proiettivo sopra un corpo qualsiasi e questioni di ampliamento. Rend. Sem. Mat. Univ. Padova **24**, 123–141 (1955).

Soient K un corps non nécessairement commutatif, $P_n(K)$ un espace projectif (droit) de dimension n sur K, et H un hyperplan de $P_n(K)$. L'ensemble $P_n(K)-H$ peut être muni de structures naturelles d'espace affine. Il y a unicité si et seulement si K est commutatif. La non-unicité dans le cas non-commutatif est liée au fait que, dans $P_n(K)$, la donnée de $n+1$ points base et d'un point unité ne détermine pas de façon unique un système de coordonnées projectives. Etude de l'extension de K en K' dans le cas non commutatif: effet sur les systèmes de coordonnées et sur le groupe des transformations projectives. *P. Samuel* (Clermont-Ferrand).

Böhm, Wolfgang. Die Fadenkonstruktionen der Flächen zweiter Ordnung. Math. Nachr. **13**, 151–156 (1955).

Verfasser hatte in seiner Dissertation [Tech. Univ. Berlin, 1953] die Maxwell-Staudesche Theorie der Fadenkonstruktion eines Ellipsoides für beliebig viele Dimensionen auch für nichteuklidische Räume induktiv behandelt. Als Beweismittel wird der Satz von Ivory verwendet: die vier Raumdiagonalen eines von drei Paaren konfokaler Flächen gebildeten Quaders sind gleich lang. Das Verfahren ist für zwei Dimensionen wohlbekannt. Aber bereits die Anwendung auf den dreidimensionalen Fall scheint in der Literatur zu fehlen. Verfasser gibt daher in der vorliegenden Arbeit eine ausführliche Diskussion des dreidimensionalen Falles, der durch instruktive Figuren erläutert wird. *M. Pinl* (Köln).

Sauer, Robert. Darboux-Kranz verknickbarer Viereck-sgitter. Arch. Math. **6**, 180–184 (1955).

An analog to the configuration of twelve surfaces known as crown of Darboux [Leçons sur la théorie générale des surfaces, partie IV, Gauthier-Villars, Paris, 1896, pp. 48–86] is given for deformable quadrilateral networks.

L. A. Santaló (Buenos Aires).

See also: Stöcker, p. 10.

Convex Domains, Extremal Problems, Integral Geometry

★ **Alexandrow, A. D. Die innere Geometrie der konvexen Flächen.** Akademie-Verlag, Berlin, 1955. xvii +522 pp. DM 38.50.

Translation of the author's Vnutrennyaya geometriya vypuklyh poverhnostel [Gostehizdat, Moscow, 1948; MR **10**, 619]. Minor errors in the text have been corrected and the author has added an Appendix (pp. 487–520) surveying results obtained since 1947.

Grotemeyer, K. P. Die eindeutige Bestimmung einer Klasse von offenen, vollständigen Flächen durch die Metrik. Math. Z. **52**, 17–22 (1955).

Result and method (namely that of Herglotz) are the same as in A. V. Pogorelov, Dokl. Akad. Nauk SSSR (N.S.) **94**, 21–23 (1954) [MR **16**, 162]. (The present paper was sent to the publisher just about when Pogorelov's paper appeared.) It is interesting to note that A. D. Alexandrov states on page 488 in an appendix to the book reviewed above that the methods used by Pogorelov for the case of bounded convex surfaces, can be used to eliminate all differentiability hypotheses in the present case of unbounded convex surfaces with total curvature 2π. *H. Busemann* (Los Angeles, Calif.).

Sandgren, Lennart. A correction to "On convex cones". Math. Scand. **3**, 170 (1955).

See Math. Scand. **2**, 19–28 (1954); MR **16**, 395.

Tvrdá, Tatiana. On the sum of convex configurations. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4**, 218–226 (1954). (Slovak)

Sind die Vektoren $\vec{OA}$. $\vec{O^1A}$, $\vec{O^2A}$ einer euklidischen Ebene durch die Beziehung $\vec{OA}=\vec{O^1A}+\vec{O^2A}$ verknüpft, so bezeichnet die Verfasserin den Punkt A als (kommutative und assoziative) „Summe" der Punkte ${}^1A$ und ${}^2A$:

$$A={}^1A+{}^2A={}^2A+{}^1A; \quad ({}^1A+{}^2A)+{}^3A={}^1A+({}^2A+{}^3A)$$

und beweist die folgenden Sätze: (1) Ist a eine beliebige Gerade durch den punkt O, die zu einer zweiten Geraden u (der gleichen Ebene) nicht parallel ist (u geht auch nicht durch O) und sind ${}^1A$. ${}^2A$ zwei eigentliche Punkte der Geraden a, die weder auf u noch auf O fallen, so gilt die Beziehung

$$\overline{O^1A}\cdot\overline{O^2A}=\overline{OA^2},$$

worin $\bar{A}$ den Schnittpunkt der Geraden a und u bezeichnet. Sind dann ${}^1a$ und ${}^2a$ zwei weitere Geraden durch die Punkte ${}^1A$ und ${}^2A$ parallel zu u, so ist der „Summenpunkt" eines beliebigen eigentlichen Punktes der Geraden ${}^2a$ und eines beliebigen eigentlichen Punktes der Geraden ${}^1a$ ein uneigentlicher Punkt. (2) Es sei ${}^1A$ ein beliebiger Punkt außerhalb der Geraden u und ${}^2a$ eine solche Gerade, daß ihre Punkte mit ${}^1A$ uneigentliche Punktsummen haben. Ferner sollen die Punkte ${}^2M$, ${}^2N$ ein Intervall bestimmen, welches mit ${}^2a$ einen Randpunkt gemein haben kann, aber keinen Punkt mit der Geraden u. Dann erzeugen die Punktsummen des Punktes ${}^1A$ mit allen Punkten des Segmentes $\overline{{}^2M{}^2N}$ wieder ein Segment, dessen Randpunkte ${}^1A+{}^2M$ und ${}^1A+{}^2N$ sind. (3) Seien $\overline{{}^1A{}^1B}$ und $\overline{{}^2A{}^2B}$ zwei Intervalle, die keinen Punkt mit u gemeinsam haben. Ferner möge die Punktmenge ${}^2\Phi$ mit dem Segment $\overline{{}^2A{}^2B}$ höchtens dessen Randpunkte ${}^2A$, ${}^2B$ gemeinsam haben zum Abschnitt $\overline{{}^1A{}^1B}$ jedoch punktfremd sein. Dann bilden die Punktsummen aller Punkte des Segmentes $\overline{{}^1A{}^1B}$ und aller Punkte des Abschnittes $\overline{{}^2A{}^2B}$ eine konvexe Figur. (4) Sind ${}^1\Phi$ und ${}^2\Phi$ zwei konvexe Figuren, die keinen Punkt der Geraden u enthalten und ist die Menge ${}^*\Phi$ (die mit ${}^2\Phi$ höchstens Randpunkte gemeinsam hat) zur Menge ${}^1\Phi$ komplementär, so ist die Punktsumme Φ der Figuren ${}^1\Phi$, ${}^2\Phi$ konvex.

M. Pinl (Köln).

Besicovitch, A. S. On limits of the area of a polygon inscribed in a simple closed curve. Quart. J. Math. Oxford Ser. (2) **6**, 135–142 (1955).

Let G be the interior domain determined by a positively oriented simple closed plane curve C. The author shows that, for a real number a to be a limit of the area-integrals $\int(xdy-ydx)/2$ of simple closed inscribed polygons tending to C, it is necessary and sufficient that $|G|\leq a\leq|G+C|$. An example is given in which the corresponding limits, for inscribed polygons not restricted to be simple, consist of all reals from $-\infty$ to $+\infty$. *L. C. Young* (Madison, Wis.).

Schäffer, Juan J. Minimum figure covering points of a lattice. II. Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist. **2**, 173–199 (1955). (Spanish. English summary)

A stronger form of the principal result is found in the author's paper, Math. Ann. **129**, 265–273 (1955) [MR **16**, 1145] which was submitted after the present paper. In addition, there are some results on convex covering sets with minimal perimeter. *H. Busemann.*

Masotti, Arnaldo. Questioni isoperimetriche nella fisica matematica. Rend. Sem. Mat. Fis. Milano **24** (1952–53), 3–33 (1954).

The author gives a survey of the classical isoperimetric inequalities and of the more recent results of similar character contained in the book of Pólya and Szegö [Isoperimetric inequalities in mathematical physics, Princeton, 1951; MR **13**, 270]. *G. Szegö.*

Differential Geometry

★**Biernacki, Mieczysław. Geometria różniczkowa. Część druga. [Differential geometry. Part 2.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1955. 248 pp. zł. 30.

[For part 1 see MR **16**, 511.] This second volume is devoted to the theory of surfaces. The first two chapters (III, IV) deal with the two fundamental forms. In the next chapter (V) the author introduces the third fundamental form, the theory of privileged curves on a surface (asymptotic curves, curves of main curvature and so on) and spherical mapping. The fundamental equations of the theory of surfaces are in Chapter VI. Chapter VII deals mainly with properties of curves and surfaces which depend on the first fundamental form. Special surfaces (ruled, minimal, Monge's, Joachimsthal's, etc.) are studied in the last chapter. The author confines himself to the classical methods which he presents very clearly. A great number of problems (with solutions) makes of this volume a very attractive textbook. *V. Hlavatý.*

Egloff, Werner. Ein geometrischer Beweis eines Satzes von Axel Schur. Arch. Math. **6**, 281–283 (1955).

A short geometrical proof of the following theorem of A. Schur [W. Blaschke, Vorlesungen über Differentialgeometrie, vol. I, 3rd ed., Springer, Berlin, 1924, p. 61] is given: If an open, plane, convex curve is deformed in such a way that the arc length and the curvature are preserved, then the chord which unites the end points increases. *L. A. Santaló* (Buenos Aires).

Brauner, H. Erzeugung eines gleichseitigen hyperbolischen Paraboloides durch Bewegung einer gleichseitigen Hyperbel. Arch. Math. **6**, 330–334 (1955).

Da die auf einer Quadrik liegenden Kegelschnitte zu je ∞^1 kongruent sind, ist es immer möglich sie als Bewegfläche eines Kegelschnittes zu erzeugen [vgl. Brauner, Monatsh. Math. **59**, 45–63 (1955); MR **16**, 1050]. Verf. betrachtet den Spezialfall der Erzeugung eines gleichseitigen hyperbolischen Paraboloids durch Bewegung einer gleichseitigen Hyperbel. Die Bahnkurven sind rationale Raumkurven vierter Ordnung, die eine Asymptote der Hyperbel erzeugt eine Ebene, die andere ein Konoid sechster Ordnung, die Hyperbelebene umhüllt einen Zylinder dessen Spurkurve ein Malta-Kreuz ist. Die Axoiden der Bewegung sind Drehzylinder. *O. Bottema* (Delft).

Voss, K. Eine Bemerkung über die Totalkrümmung geschlossener Raumkurven. Arch. Math. **6**, 259–263 (1955).

Let F be a continuously curved closed surface with the Gaussian curvature K and the area element dA. Considering the spherical image of those parts of F which lie on the convex hull of F, the author obtains (1) $\int_{K>0} KdA\geq 4\pi$. Now let C be a twice continuously differentiable closed skew curve with the arc length s and the curvature $\varkappa(s)$. If $r>0$ is sufficiently small, the spheres of radius r about the points of C envelop a canal surface F which satisfies (2) $\int_{K>0} KdA=2\oint|\varkappa|ds$. Thus (1) implies Fenchel's inequality: (3) $\oint|\varkappa|ds\geq 2\pi$. The same method yields Fáry's result $\oint|\varkappa|ds\geq 4\pi$ if C is knotted. Equality in (3) implies equality in (1). All the points of F with $K>0$ then lie on the convex hull of F. This readily implies that C is plane and convex. (2) is shown to be equivalent to a formula of integral geometry used by Fenchel in a proof of (3) [cf. W. Fenchel, Bull. Amer. Math. Soc. **57**, 44–54 (1951); MR **12**, 634]. *P. Scherk* (Saskatoon, Sask.).

Wunderlich, W. Kreise als Doppelloxodromen. Arch. Math. **6**, 230–242 (1955).

A loxodrome is a curve which intersects the planes through a fixed axis a at a constant angle. The circle l is a loxodrome with the axis a if and only if the centers of the two null-spheres through l lie in the minimal planes through a. This leads to a simple discussion of the ∞^2 circles which are loxodromes with two given axes. Exactly eight of them form given angles with the planes through these axes.

The author then studies l's which intersect the pencil of the spheres through a given real circle c at a constant angle δ. The centers of the null-spheres through l lie on the null-spheres of the pencil and vice-versa. The l's with given c and δ and the set of those l's are discussed which intersect two such pencils at constant angles. In general, this set will contain exactly 32 circles which form given angles with these pencils. Finally, a non-euclidean generalization is outlined. *P. Scherk* (Saskatoon, Sask.).

Hirakawa, Junkō. On the relative minimal surface. J. Math., Tokyo **1**, 67–70 (1953).

A fixed surface $E(f)$ is considered as the unit surface, and an arbitrary surface (Sx) is parameterized in such a way that tangent planes are parallel at corresponding points $x(u, v)$ and $f(u, v)$; for example, this is true if

$$x\equiv[\tfrac{1}{2}a(u+v)+\alpha,\ \tfrac{1}{2}b(u-v)+\beta,\ \tfrac{1}{2}uv],$$
$$f\equiv[\tfrac{1}{2}a(u+v)+\gamma,\ \tfrac{1}{2}b(v-u)+\delta,\ \tfrac{1}{4}(u^2+v^2)],$$

where $\alpha, \beta, \gamma, \delta$ are constants. The vector $-f$ is called the r-surface normal, the first r-fundamental quadratic form is taken to be $dS^2=Q^2(dx)^2$, where $Q(u, v)$ is the function

of supporting planes for $E(f)$, and so on for the other fundamental quantities [cf. Hirakawa, Jap. J. Math. **17**, 347–400 (1941); MR **7**, 394]. Surfaces for which the first variation of the r-surface area vanishes are called r-minimal surfaces, and it is shown that a surface is r-minimal if and only if its r-mean curvature vanishes identically. For example, the foregoing surface $S(x)$ is r-minimal. Additional analogues of classical properties of minimal surfaces are established for r-minimal surfaces.

E. F. Beckenbach (Los Angeles, Calif.).

Moisil, Gr. C. Une caractérisation des transformations pseudo-conformes. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7**, 5–17 (1955). (Romanian. Russian and French summaries)

Various formal criteria for the pseudoconformality of a differentiable mapping in a real four-space. *L. Bers.*

Rembs, Eduard. Verbiegbarkeit konvexer Kalotten mit zylindrischen und konischen Randstreifen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 315–320 (1955).

Given a convex surface C homeomorphic to a disc. Suppose its boundary strip is cylindrical [conical]; i.e., the tangent planes of C along its boundary curve are parallel to a given straight line [pass through a given point]. Theorem 1α [1β]: C is rigid under infinitesimal isometric deformations which keep the boundary strip cylindrical [conical]. Theorem 2α [2β]: Two isometric C's with cylindrical [conical] boundary strips are congruent or symmetric. Theorem 2β is new, while the proofs of the three other theorems are simplified considerably. An error in the second proof of Theorem 1β is corrected. References: Grotemeyer, Math. Z. **58**, 272–280 (1953), Th. 1α, 2α [MR **15**, 158]; Rembs, Math. Z. **56**, 271–279 (1952), Th. 1α; Math. Ann. **127**, 251–254 (1954), Th. 1β; Arch. Math. **4**, 366–368 (1953), Th. 1β [MR **14**, 901; **15**, 740, 554; the review of the last paper was erroneous; the above references indicate how it should be corrected].

P. Scherk (Saskatoon, Sask.).

Bompiani, Enrico. Deformazioni di superficie di uno spazio euclideo con linee e striscie rigide. Matematiche, Catania **9**, 154–175 (1954).

A surface S_2 in the euclidean space E_3 with a fixed line C is undeformable (isometrically) unless C is an asymptotic line of S_2. This classical theorem is generalized in different ways: a) If a surface S_2 in E_4 admits a deformation which preserves a given strip attached to a curve C, then the osculating 3-spaces to C contain the corresponding tangent planes to S_2; b) a deformation which preserves the arc lengths and the curvatures of order $\leq \nu$ of the curves of S_2 is called a deformation of order ν. Then a deformation of order ν of a surface S_2 in $E_{2\nu+1}$ which preserves a line C is only possible if C is an ordinary asymptotic line of S_2. The generalization to spaces E_k ($k>2\nu+1$) is also considered and some particular cases are analyzed. *L. A. Santaló* (Buenos Aires).

Ritter, Robert. Charakterisierung der Baronischen Klassen von Biegungsflächen. Arch. Math. **6**, 311–319 (1955).

Let P be a point on a Goursat surface, G_λ. Let O be a fixed point. Then the midpoint of the principal centers of curvature on the surface normal at P is λ times as far from the tangent plane as O is, λ a constant. The Weingarten equations make it possible to associate with each G_λ a Baroni surface, B_λ, using a vector quadrature. Conversely, each B_λ can be shown to be associated with a G_λ by this quadrature.

The principal new result of this paper is the theorem that a surface is a B_λ, $\lambda\neq -1$, if and only if $\varrho=K^{-\frac{1}{2}}$ satisfies three invariant conditions:

$$\varrho\Delta_2\varrho=-\tfrac{1}{2}\Delta_1\varrho-4\lambda,$$
$$\varrho\Delta(\varrho,\Delta_1\varrho)=-[\Delta_1\varrho+8(\lambda+1)](\Delta_1\varrho-4),$$
$$\varrho\Theta(\varrho,\Delta_1\varrho)=16(\lambda+1)(\Delta_1\varrho-4)^{\frac{1}{2}}.$$

The case $\lambda=-1$ is simpler and leads to the conditions $\Delta_1\varrho=4$, $\varrho\Delta_2\varrho=2$. The author also gives a canonical line element for B_λ, $\lambda\neq -1$, using as parameter curves the lines of constant pseudoradius $\varrho=K^{-\frac{1}{2}}$ and their orthogonal trajectories. *A. Schwartz* (New York, N.Y.).

Blanuša, Danilo. Le plongement isométrique de la bande de Möbius infiniment large euclidienne dans un espace R_5. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 7–10 (1954).

In R_3 a Möbius band can be imbedded only if it is sufficiently narrow. It is proved in this paper that it is possible to imbed isometrically in R_5 a Möbius band B of infinite width. First a surface of revolution in R_3 is constructed on which the band can be wrapped so that the (closed) horizontal and the (infinite) vertical generators of B are sent into orthogonal families, and so that the mapping is a homeomorphism for all points except one horizontal generator, which is sent into a single point. Then a mapping into R_2 is constructed, which sends the closed generators of B into concentric circles, and the others into the rays through the center. After these preparations the isometric mapping of B into $R_3\times R_2$ is easily constructed. The imbedding contains an undetermined function f. By a suitable choice of f, B becomes an algebraic surface in R_5, given by

$$x_1(x_2^2+x_3^2)=4x_1-1,\quad 4x_1(x_4^2+x_5^2)=1,$$
$$(4x_1-1)^2x_5^2=x_1x_2^2x_3^2.$$

A. Nijenhuis (Chicago, Ill.).

Blanuša, Danilo. Isometric imbedding of a Euclidean infinitely wide Möbius band in the space R_5. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 9–15 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Blanuša, Danilo. Le plongement isométrique de la bande de Möbius infiniment large euclidienne dans un espace sphérique, parabolique ou hyperbolique à quatre dimensions. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 19–23 (1954).

By projecting the infinitely wide Möbius band B, imbedded in R_5 [see the second preceding review] on a sphere S_4 around the origin of R_5, and by subsequently modifying the imbedding somewhat, the author succeeds in imbedding B isometrically in S_4. If the length of B is L, the radius R of S_4 must satisfy $R\geq L/2\pi$. Since in S_4 no two diametric points both belong to B, one also has an imbedding of B into El_4. By taking in S_4 the limit $R\to\infty$ one obtains an imbedding of B into R_4 (not as an algebraic surface), and by formal extension of the imbedding formulas for S_4 one gets an isometric imbedding in the hyperbolic space H_4. In all cases B is mapped into a bounded subset of the space in question.

A. Nijenhuis (Chicago, Ill.).

Blanuša, Danilo. Isometric imbedding of a Euclidean infinitely wide Möbius band in a four-dimensional spherical, parabolic or hyperbolic space. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 49–55 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Blanuša, Danilo. Plongement isométrique de l'espace hyperbolique à n dimensions à distance finie d'un point dans l'espace de Hilbert. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 25–30 (1954).

Starting from an imbedding of the hyperbolic space H_2 in Hilbert space R_∞ [special case of H_n in R_∞, cf. Monatsh. Math. **57**, 102–108 (1953); MR **15**, 61] a modified imbedding is constructed so that H_2 lies in a bounded subset. The formulas are then generalized for general H_n.

A. Nijenhuis (Chicago, Ill.).

Blanuša, Danilo. Isometric imbedding in Hilbert space of an n-dimensional hyperbolic space with finite distance from a point. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 57–64 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Blanuša, Danilo. Immersion du cylindre et du plan euclidiens dans des espaces sphériques. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 161–166 (1954). (Serbo-Croatian summary)

In a previous paper [same Glasnik **9**, 15–25 (1954); MR **16**, 515] the author gave an isometric imbedding of a torus with arbitrary fundamental parallelogram in the sphere S_3 and the elliptic space El_3. A modification of the imbedding formulas makes it possible to imbed isometrically in S_3 an infinitely long cylinder. *A. Nijenhuis.*

Helfenstein, H. G. Conformal maps with least distortion. Canad. J. Math. **7**, 306–313 (1955).

In a conformal mapping the ratio of two corresponding line elements is a function of position in general. This paper considers the problem of minimizing the mean quadratic deviation of this ratio from a constant. Let D be a finite, simply connected domain with more than one boundary point on a surface. Find an isothermic parameter system mapping D on a plane schlicht domain D', which in turn can be transformed conformally on the interior of the unit circle U of a complex z-plane. If $z=x+iy$, then x and y are called a normal system of isothermic parameters on D, uniquely determined if an arbitrary point M of D is selected to correspond to $z=0$ and an arbitrary direction through M to correspond to the positive X axis. For the line element of D we have $ds=|dz|/\gamma(z)$. Every other conformal map of D can be obtained by means of a regular analytic function $w=f(z)$ defined in U with non-vanishing derivative. The distortion becomes $|dw|/ds=e^{\phi+\theta}$, where $\theta=\log\gamma(z)$, $\phi=\log|f'(z)|$. Assume for D that θ is twice differentiable with derivatives bounded in $|z|<1$. Then the author determines a harmonic function ϕ and a constant C such that $I=\int\int_U(\phi+\theta-C)^2dxdy$ becomes a minimum. Explicit formulas for $f(z)$ are given, as is an integral representation for $\log f'(z)$. As a simple example, for a circular domain on a sphere the conformal mapping with least distortion in this sense is its symmetrical stereographic projection. One unsolved problem mentioned is concerned with the best choice of the point M which is to serve as the "center of the map". *A. Schwartz* (New York, N.Y.).

MacQueen, M. L. The quadric of Wilczynski. J. Tennessee Acad. Sci. **28**, 285–286 (1953).

A surface of a projective space whose asymptotes are taken as coordinates is determined by the equations

$$x_{uu}=px+\theta_u x_u+\beta x_v,\quad x_{vv}=gx+\gamma x_u+\theta_v x_v \qquad (\theta=\log\beta\gamma).$$

The lines $l_1(a,b)$, $l_2(a,b)$ are reciprocal at an ordinary point x of the surface if the line l_1 joins the points x and $y=-ax_u-bx_v+x_{uv}$ and the line l_2 joins the points $\varrho=x_u-bx$, $\sigma=x_v-ax$. The lines $\varrho\varrho_v$ and $\sigma\sigma_u$ intersect the line l_1 respectively in the points

$$(ab-b_v)x+y,\quad (ab-a_u)x+y.$$

The point harmonically conjugate to the point x with respect to these two points has the local coordinates

(1) $x_1=ab-\frac{1}{2}(a_u+b_v)$, $x_2=-a$, $x_3=-b$, $x_4=1$.

Theorem: If the line l_1 is the axis of a pencil of osculating planes of hypergeodesics for a pencil of conjugate nets of the surface, then the locus of the point (1) is the quadric of Wilczynski. *N. V. Laktanova* (RŽMat. **1955**, no. 3939).

Vaccaro, Giuseppe. Sulle calotte superficiali dello spazio a tre dimensioni. Rend. Mat. e Appl. (5) **14**, 525–532 (1955).

Given a surface of the 2nd order (or cap) σ_2 in a projective space S_3 and a conic C (generically situated with respect to σ_2), each generic linear element E_2 determines a conic $\gamma\supset E_2$ and incident to C in two points. The elements (with the same center of σ_2) of the 3rd (or 4th) order of these conics γ constitute a prolongation of 3rd (or 4th) order of σ_2 with respect to C. It is proved that there are ∞^4 3rd order prolongations of σ_2 (∞^3 of them belonging to quadrics).

A 3rd order cap $\sigma_3\supset\sigma_2$ has ∞^1 E_3 in common with a given prolongation of σ_2; if the prolongation belongs to a quadric, these elements E_3 are distributed in 3 pencils. A 4th-order cap $\sigma_4\supset\sigma_2$ has 10 elements E_4 in common with a prolongation of σ_2; if the prolongation is determined by the absolute of a Euclidean space, a metric theorem already given by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 35–41 (1954); MR **16**, 67] is obtained. *E. Bompiani* (Rome).

Vaona, Guido. Varietà caratteristiche di una trasformazione puntuale e varietà quasi-asintotiche. Boll. Un. Mat. Ital. (3) **10**, 32–42 (1955).

Una trasformazione puntuale T tra due spazi S_r e S_r' si può ottenere proiettando opportunamente i punti di una V_r di un S_m ($m\geq 2r+1$) per i due S_r. Se la V_r contiene per ogni generico punto una o più calotte h-dimensionali quasi asintotiche $\sigma_{1,2}$ di specie massima, la trasformazione T possiede almeno ∞^{h-1} direzioni caratteristiche, per ogni coppia generica di punti corrispondenti, costituenti uno o più S_h.

In generale il viceversa non è vero e la conoscenza delle trasformazioni che posseggono iperpiani di rette caratteristiche per ogni punto [Vaona, Univ. e Politec. Torino. Rend. Sem. Mat. **12**, 195–238 (1953); MR **15**, 741] permette all'A. di determinare le V_k ($k>3$) che posseggono ∞^k calotte $(k-1)$-dimensionali quasi asintotiche $\sigma_{1,2}$ di specie massima. *C. Longo* (Roma).

Mastrogiacomo, Pasquale. Trasformazioni puntuali tra spazi proiettivi osculabili con trasformazioni quadratiche di seconda specie particolari. Ricerca, Napoli **5**, no. 1–2, 27–32, no. 3, 40–49 (1954).

A quadratic transformation of the second type is a

projective correspondence between the planes of an S_3 and the quadrics of an S_3', through a line and three generic points. It is known that there are 8 types of this transformation. Besides, it is known that a point-transformation admits in a generic pair of corresponding points 7 characteristic lines which are the basis lines of a linear system ∞^2 of cubic cones. The author determines the 8 types of the preceding linear systems which characterize the 8 types of point-transformations which may be approximated up to the second order by the 8 types of quadratic transformations. *C. Longo* (Rome).

Speranza, Francesco, Sulle trasformazioni puntuali fra spazi proiettivi sovrapposti. Boll. Un. Mat. Ital. (3) **10**, 61–68 (1955).

Una trasformazione puntuale T tra due spazi lineari S, sovrapposti determina in una coppia regolare A, $\bar{A}$ di punti corrispondenti una omografia Ω. Supposto A ed $\bar{A}$ distinti ed Ω generale, l'Autore determina un riferimento intrinseco dipendente dall'intorno del 1° ordine. L'Autore esamina poi il caso che la Ω sia una prospettività. *C. Longo* (Roma).

Su, Buchin. The step point of a curve in a projective space. Acta Math. Sinica **4**, 33–79 (1954). (Chinese. English summary)

An English version of this paper appeared in Acta Sci. Sinica **3**, 107–151 (1954); MR **16**, 854.

Balazs, N. L. The resolution of four-dimensional vector fields and tensor fields, and its application to electrodynamics. Canad. J. Phys. **33**, 235–240 (1955).

In Euclidean E_4 a vector field V_i (assumed to vanish sufficiently rapidly at infinity) is decomposed into $V_i=V_i^1+V_i^2$ where V_i^1 is irrotational ($V^1_{i,k}-V^1_{k,i}=0$) and V_i^2 solenoidal ($V^2_{i,i}=0$), the fields V_i^1 and V_i^2 being expressed as integrals over E_4. Similarly a skew-symmetric tensor h_{ik} is decomposed into an irrotational part h^1_{ik} ($\hat{h}^1_{ik,k}=0$, where $\hat{h}^1_{ik}$ is the dual of h^1_{ik}) and a solenoidal part h^2_{ik} ($h^2_{ik,k}=0$). The formulae are

$$h^1_{ik}=a_{k,i}-a_{i,k}, \quad a_i=(2\pi)^{-2}\int h_{ik,k}R^{-2}d^4\xi,$$

with a similar formula for $\hat{h}^2_{ik}$; R denotes distance in E_4. The decompositions are unique, except for trivial additive constants. The author passes from E_4 to Minkowskian M_4 by making the fourth coordinate complex and integrating round singularities in the manner of Herglotz and Sommerfeld. But this is not carried out explicitly, the integrals being left in the form appropriate to E_4. For the electromagnetic tensor of a Maxwellian field, the irrotational part arises from the electric current, and the solenoidal part vanishes since magnetic current is absent. The author comments on the normalising condition $A_iA_i=$const. imposed on the 4-potential by P. A. M. Dirac [Proc. Roy. Soc. London Ser. A **209**, 291–296 (1951); MR **13**, 893]. [The decomposition of a skew-symmetric tensor into irrotational and solenoidal parts has been used by B. Finzi, Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, vol. II, Edizioni Cremonese, Roma, 1953, pp. 501–506; MR **15**, 185. See also B. Finzi, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 378–382, 477–480 (1952); **13**, 211–215 (1952); MR **15**, 665, 666; A. M. Pratelli, Ann. Scuola Norm. Sup. Pisa (3) **7**, 161–203 (1954); MR **16**, 652. For explicit formulae involving retarded values, see E. Durand, C. R. Acad. Sci. Paris **239**, 751–753 (1954); MR **16**, 546. In the paper under review it is not pointed out that the uniqueness theorems do not pass over from E_4 into M_4. The 4-dimensional integration in E_4 becomes a 3-dimensional integration in M_4 over a complete null cone, and, mathematically, the past and future sheets can be given any relative weights we like.] *J. L. Synge* (Dublin).

Santaló, L. A. On a theorem of Holditch and analogues in non-Euclidean geometry. Math. Notae **14**, 32–49 (1954). (Spanish)

Many known formulas of n-dimensional non-Euclidean geometry are derived, including the Gauss-Bonnet Theorem for even dimensions. Then the analogue to Holditch's Theorem in two dimensions [previously obtained by Vidal Abascal and Rodeja, Collect. Math. **5**, 331–337 (1952); MR **15**, 740] is derived, as well as the following analogue to a Euclidean theorem by Kampe: If a non-Euclidean plane is moved (in itself) such that it returns to its initial position, then the loci of points whose orbits enclose equal areas are concentric circles. This is extended to motions depending on 2 parameters. The principal result is the following extension of Holditch's Theorem to a non-Euclidean space R of an arbitrary even dimension n: Under a continuous motion of R let $n+1$ collinear points A_i ($i=0, \cdots, n$) describe closed hypersurfaces bounding volumes V_i; then

$$\sum_{i=0}^{n}\{\tfrac{1}{2}j\omega_n-K^{n/2}V_i\}\Big(\prod_{j\neq i}\sin K^{\frac{1}{2}}(s_i-s_j)\Big)^{-1}\}=0,$$

where j is the number of times the direction of the line A_0A_i traverses the unit sphere, $\omega_n=2\pi^{n+1}/\Gamma(\frac{1}{2}(n+1))$, K is the curvature of the space and s_i is the distance of A_i from A_0. *H. Busemann* (Los Angeles, Calif.).

Barthel, Woldemar. Variationsprobleme der Oberflichenfunktion in der Finslerschen Geometrie. Math. Z. **62**, 23–36 (1955).

In an n-dimensional Finsler space let for every p, $0<p\leq n$, a p-dimensional area $\int F^{(p)}(x(u), \partial x/\partial u_\varrho)du_1\cdots du_p$ be defined, where $x(u)=x(u_1, \cdots, u_p)$ is a p-dimensional surface and $F^{(p)}(x, X_\varrho)\geq 0$ with equality only for linearly dependent X_ϱ and $F^{(p)}(x, U^\nu_\varrho X_\varrho)=|\,\|U^\nu_\varrho\|\,|F^{(p)}(x, X_\varrho)$. The only p considered are $p=1$, $n-1$, n, and for $p=1$, $n-1$ Legendre conditions are postulated. An invariant differentiation is defined for $(n-1)$-dimensional vector densities such that area is invariant under parallel displacement. A hypersurface is geodesic if its tangential elements go into themselves under parallel displacement along the surface. The normal curvature of a hypersurface strip is defined by using a sine function analogous to that of the reviewer's. For the normal curvatures of the strips in a surface analogs to the formulae of Olinde-Rodriques and Euler are derived, and so is the first variation of area, and thus the equation for minimal surfaces. Also, a necessary condition for solution of the isoperimetric problem is obtained.

If $F(x)$ is defined by $F^{(n)}(x, X_\nu)=F(x)\|X_1, \cdots, X_n\|$, then $F(x)$ is a scalar density. With the tangent $\dot{x}(t)$ of a curve $x(t)$ a hypersurface element $\mathfrak{p}^i(x, \dot{x})$ is associated by $F(x)\dot{x}^i=\mathfrak{p}^i(x, \dot{x})$ and the element $d^*s=f(x, \mathfrak{p})dt$, where $\mathfrak{p}_i=\|\delta_i^k, X_1^k, \cdots, X_{n-1}^k\|$ and $f(x, \mathfrak{p})=F^{(n-1)}(x, X_\varrho)$. The great arbitrariness in the definition of invariant differentiation can then be eliminated by requiring that 1) the geodesic hypersurface are minimal surfaces, 2) the dual geodesics, i.e. those curves for which the associated hypersurface elements go into themselves under parallel

displacement along the curve, are extremals for the dual length. The equations for minimal surfaces and solutions of the isoperimetric problem then take their usual forms. *H. Busemann* (Los Angeles, Calif.).

Busemann, Herbert. On normal coordinates in Finsler spaces. Math. Ann. **129**, 417–423 (1955).

The purpose of this paper is to study the restrictions placed on a Finsler space by assuming the existence everywhere of normal coordinates of class C^2. Let F be a Finsler space with normal coordinates everywhere of class C^2. Then the author proves the following theorems: If F is a two-dimensional manifold, then it is Riemannian or Minkowskian; the local Minkowskian geometries at the different points of the space F are all isometric; if the groups of rotations about a point of F in the local Minkowskian geometry is finite, then the metric is Minkowskian. The proof of these theorems rests on the fact that one may define on such a space F a parallel displacement which has all the usual properties of parallel displacement in Riemannian geometry. *L. Auslander.*

Riemannian Geometry, Connections

Ôtsuki, Tominosuke. Note on compact manifolds with non-symmetric metric connections. Math. J. Okayama Univ. **4**, 103–114 (1955).

The main concern of the author is the question under what conditions some theorems of harmonic analysis hold in case the customary Riemannian connection is replaced by a non-symmetric one, with torsion $S_{ij}{}^k$, which leaves the metric invariant.

The operators Δ, $\hat{d}$, $\hat{\delta}$, $\hat{\Delta}=-(\hat{d}\hat{\delta}+\hat{\delta}\hat{d})$ are defined for the general connection. Typical results are: 1) $\int_{V_n}\Delta\varphi d\sigma=0$ for all functions φ if and only if $S\equiv S^i{}_{k}{}^{k}{}_{|i}-2S^i{}_{k}{}^{k}S_{ir}{}^{r}=0$ (V_n compact and orientable); 2) in order that $\hat{d}\varphi=\hat{\delta}\varphi=0$ (φ is a differential form) it is sufficient that $\hat{\Delta}\varphi=0$, and $S_{ik}{}^{k}\{(\varphi\lrcorner\hat{d}\varphi)^i-(\hat{\delta}\varphi\lrcorner\varphi)^i\}=0$. Here $\varphi\lrcorner\psi$ is called the left inner product, and is defined by

$$\varphi\lrcorner\psi=\frac{1}{(p-1)!}\varphi^{i_1\cdots i_p}\psi_{ji_1\cdots i_p}dx^j.$$

There are also statements of pseudo-Killing differential forms. The work is a continuation of some papers on curvature and Betti numbers by Bochner [Ann. of Math. (2) **49**, 379–390 (1948); **50**, 77–93 (1949); MR **9**, 618; **10**, 571] and Bochner and Yano [ibid. **56**, 504–519 (1952); MR **14**, 904], and deals with more general conditions. *A. Nijenhuis* (Chicago, Ill.).

Kobayashi, Shôshichi. Des groupes linéaires irréductibles et la géométrie différentielle. Proc. Japan Acad. **30**, 934–936 (1954).

The following theorem is proved: Let V be an irreducible Riemannian space of n dimensions, σ_p the restricted holonomy group at p, and H_p the isotropy group at p; then the identity component of H_p is contained in σ_p, or alternatively (only if $n=2m$), V is pseudo-Kählerian with vanishing Ricci curvature. The dimensions d and r of σ_p and the "group" of isometries satisfy $r\leq n+d$ in the first case, and $r\leq n+d+1$ in the second, pseudo-Kählerian, case. The theorem is a consequence of a few lemmas on irreducible Lie groups G that contain a subgroup H reducible in G in the sense of Koszul. *A. Nijenhuis.*

Takano, Kazuo. On a representation of general affine paths. Tensor (N.S.) **4**, 129–134 (1955).

The author starts off with an account of some of the basic notions of non-holonomic submanifolds immersed in a space with an affine connection, and gives an expression for the projection of the normal curvature vector of a curve in the general space on a non-holonomic submanifold. By taking a particular set of non-holonomic frames, he obtains from this the left-hand side of the equations of paths in ordinary theory. *E. T. Davies.*

Ishihara, Shigeru, and Obata, Morio. On manifolds which admit some affine connection. J. Math., Tokyo **1**, 71–76 (1953).

The fiber bundle of affine connections over a manifold of dimension n has a fiber homeomorphic to Euclidean space E_{n^2}, and admits consequently a cross-section [Steenrod, Ann. of Math. (2) **43**, 116–131 (1942); MR **3**, 144]. The author shows that the condition that, p given, everywhere linearly independent vector fields v_α ($\alpha=1,\cdots,p$) with $[v_{(\alpha)}, v_{(\beta)}]=0$ be covariant constant, and that the connection be symmetric, imposes linear conditions on the connection, thus leaving the fiber of the admissible connections a contractible space. As a consequence a linear connection satisfying these conditions, exists. There is a discussion on the differentiability of the cross-section in terms of the corresponding properties of the vector fields. If M is compact and orientable, of class C^∞, and if $p=n$, then it follows that the pth Betti number of M is ${}_nC_p$. *A. Nijenhuis* (Chicago, Ill.).

Šapiro, Ya. L. Geodesic fields of directions and projective systems of paths. Mat. Sb. N.S. **36(78)**, 125–148 (1955). (Russian)

Let us consider a holonomic field of m-dimensional directions formed by m vectors $\overset{\alpha}{V}{}^a$ ($\alpha,\beta=1,\cdots,n+m$; $a,b=n+1,\cdots,n+m$) in an affinely connected space of $n+m$ dimensions A_{n+m}; then the field defines a family of m-dimensional subspaces V_m in A_{n+m}. A holonomic field of m-dimensional directions is called "geodesic", if the totality of the subspaces V_m passing through every point of any geodesic line which is not contained in any V_m is a totally geodesic subspace of $m+1$ dimensions in A_{n+m}. A necessary condition that the field of m-dimensional directions formed by $\overset{\alpha}{V}{}^a$ be geodesic is given by expressions in terms of the vectors $\overset{\alpha}{V}{}^a$, and it is shown that if such a geodesic field exists in A_{n+m}, the coefficients of connection of A_{n+m} can be reduced to $\Gamma^i{}_{\alpha\beta}=\Pi^i{}_{\alpha\beta}(x^k)+\varphi_{(\alpha}\delta^i{}_{\beta)}$, $\Pi^i{}_{\alpha\beta}=0$ ($i=1,\cdots,n$), and conversely. All such A_{n+m} form a class contained in the general one. A special one of this class is V. F. Kagan's subprojective space of n dimensions [Trudy Sem. Vektor. Tenzor. Anal. **1**, 12–101 (1933)], which contains a linear line system of an $(n-1)$-dimensional euclidean space. If a system of paths in an n-dimensional manifold can be supplemented by others so that it defines with the given system a complete system of geodesic lines in an affinely connected $(n+m)$-dimensional space, then the system of paths is said to be "imbedded" (projectively). Making use of the concept "imbedding", it is also proved that the above stated necessary condition is sufficient. Then the obtained results are applied to a Riemannian space containing a geodesic field of directions. At the end, there are some appendices where it is shown that the results can be applied readily to metric and generalized subprojective spaces, the gener-

alized projective geometry of H. Weyl [Raum, Zeit, Materie, 4. Aufl., Springer, Berlin, 1921], and the relativistic theory of a central-symmetric gravitational field.

A. Kawaguchi (Sapporo).

Nash, John. A path space and the Stiefel-Whitney classes. Proc. Nat. Acad. Sci. U.S.A. **41**, 320–321 (1955).

In the present note, the author gives a very simple demonstration of the topological invariance of the Stiefel-Whitney classes for a differentiable manifold M. This invariance has been proved by other means by R. Thom [Ann. Sci. Ecole Norm. Sup. (3) **69**, 109–182 (1952); MR **14**, 1004]. Let X denote the space of all (parametrized) paths $x: I \to M$ such that $x(t) \neq x(0)$ for all $t > 0$. Then X is a fiber space over M relative to the projection $p: X \to M$ defined by $p(x) = x(0)$ for each x in X. One may regard M as provided with a Riemannian metric and assume that there is a unique shortest geodesic segment joining any two points of distance not exceeding 1. Then the tangent bundle T of M can be regarded as formed by the geodesic paths of length 1 parametrized by the arc length and hence is a subspace of X. The author sketches a proof that T is a fiber deformation retract of X. This implies that the Stiefel-Whitney classes of M do not depend on the differential structure of M.

S. T. Hu (Athens, Ga.).

Dedecker, Paul. Jets locaux, faisceaux, germes de sous-espaces. Bull. Soc. Math. Belg. **6** (1953), 97–125 (1954).

The author develops an axiomatic theory of great generality designed to deal with the step from "local" to "global" that includes the notions of jet [Ehresmann, Colloque de topologie et géometrie différentielle, Strasbourg 1952; no. 11, Bibliothèque Nat. et Univ. Strasbourg, 1953; MR **15**, 828], sheaf (faisceau) [J. Leray, J. Math. Pures Appl. (9) **29**, 1–139 (1950); MR **12**, 272; and Séminaire H. Cartan, Ecole Norm. Sup., 1950/51, 51/52, 53/54; MR **14**, 670; **16**, 233], germ of analytic subset [H. Cartan, loc. cit.] and germ of subspace [Dedecker, Bull. Math. Soc. Belg. **1952**, 26–43 (1953); MR **15**, 547]. This necessitates a number of new definitions, such as local type of maps (system of maps of sets with a given kind of local structure into sets of some other class, closed under composition with isomorphisms, and under formation of "union" of maps), bilocal type of maps, bilocal type of immersion, etc, and generalizations of many of the concepts used by the earlier authors. As an application the author shows how in his theory one can identify the principal fiber bundles over a given base B (with a given structure group) with the element of a one-dimensional cohomology "group" of B with coefficients in a certain sheaf of (non-abelian) groups over B. *H. Samelson.*

Ehresmann, Charles. Les prolongements d'un espace fibré différentiable. C. R. Acad. Sci. Paris **240**, 1755–1757 (1955).

Continuation of earlier notes [same C. R. **239**, 1762–1764 (1954); **240**, 397–399 (1955); MR **16**, 625]. The notion of fiber space is generalized by letting a groupoid act on a space (suggested by the groupoid of isomorphisms between the fibers of a fiber bundle), subject to suitable axioms. Concepts of principal groupoid, prolongement (non-holonomic and semi-holonomic) of the groupoid if the action is differentiable, prolongement of the fiber spaces; discussion of the ordinary fiber bundles in the new frame work. *H. Samelson* (Ann Arbor, Mich.).

Complex Manifolds

Grauert, Hans. Charakterisierung der holomorph vollständigen komplexen Räume. Math. Ann. **129**, 233–259 (1955).

Ce mémoire important contient des théorèmes profonds, de démonstration difficile. Rappelons qu'une variété de Stein („holomorph vollständige Mannigfaltigkeit") est une variété analytique-complexe X satisfaisant aux quatre conditions: (i) X est réunion dénombrable de compacts; (ii) si $x \in X$, $x' \in X$, $x \neq x'$, il existe f holomorphe dans X telle que $f(x) \neq f(x')$; (iii) pour chaque $x \in X$ il existe un système fini de f_i holomorphes dans X, qui constituent des coordonnées locales au point x; (iv) X est „holomorphiquement convexe" (i.e.: pour tout compact $K \subset X$, l'ensemble des x tels que toute f holomorphe dans X satisfasse à $|f(x)| \leq \sup_{y \in K} |f(y)|$, est compact).

Soit maintenant X un „espace complexe" [au sens de Behnke-Stein, cf. le mémoire analysé ci-dessous]. Les conditions (i), (ii) et (iv) conservent un sens; l'auteur modifie (iii) comme suit: (iii') pour chaque $x \in X$ il existe un système fini de f_i holomorphes dans X, qui définissent une réalisation d'un voisinage de x (dans X) comme sous-ensemble analytique normal d'un ouvert d'un espace numérique complexe C^n.

Un espace complexe satisfaisant à (i), (ii), (iii') et (iv) est dit „holomorphiquement complet". Ces espaces sont importants, car la théorie des faisceaux cohérents de Oka-Cartan-Serre est valable pour eux. L'auteur se propose de remplacer les conditions (i), (ii) et (iii') par une condition plus faible; or (iii') implique la condition (K): Pour tout $x \in X$, il existe une application analytique f de X dans un espace C^k (k dépendant de x), non dégénérée au point x (i.e. x est point isolé de $f^{-1}(f(x))$). Remarque: cette condition implique que l'espace complexe X est un „espace analytique général" au sens de H. Cartan.

Théorème fondamental: Si un espace complexe X, irréductible, satisfait à la condition (K) et est holomorphiquement convexe (condition (iv)), X est holomorphiquement complet.

On prouve d'abord (Satz 8): si X irréductible satisfait à la condition (K), X est réunion dénombrable de compacts. D'autre part (Satz 12): si X irréductible, de dimension (complexe) n, satisfait à la condition (K), il existe une application analytique $f: X \to C^n$ qui n'est dégénérée en aucun point (ce résultat est nouveau même dans le cas où X est une variété de Stein).

Il est impossible de donner des indications, même sommaires, sur les démonstrations de ces théorèmes (Satz 8, 12); même prouvés, ils n'entraînent pas aussitôt le théorème fondamental, qui nécessite notamment les deux lemmes suivants, intéressants pour eux-mêmes:

Lemme 1: Soit X un espace complexe; soient n fonctions f_i holomorphes en un point $x \in X$, nulles en x, et telles que toute fonction holomorphe en x s'exprime comme fonction holomorphe des f_i; alors tout système de g_i holomorphes en x, nulles au point x, et assez voisines des f_i jouit de la même propriété.

Lemme 2: soient X et X' deux espaces complexes, et soit $f: X \to X'$ une application analytique telle que l'image réciproque de tout ensemble compact (resp. fini) de X' soit un ensemble compact (resp. fini) de X; si X' est une variété de Stein, alors X est holomorphiquement complet.

La fin de la démonstration du théorème fondamental utilise en outre le fameux „Heftungslemma" de Oka.

H. Cartan (Paris).

Grauert, Hans, und Remmert, Reinhold. Zur Theorie der Modifikationen. I. Stetige und eigentliche Modifikationen komplexer Räume. Math. Ann. **129**, 274–296 (1955).

La notion de „modification" a été introduite par Behnke et Stein [Math. Ann. **124**, 1–16 (1951); MR **13**, 644], ainsi que celle d'„espace analytique complexe" (plus brièvement: espace complexe) qui généralise la notion de variété analytique complexe, de manière à inclure des ramifications internes. Les auteurs donnent une nouvelle exposition de ces notions (on ignore si la notion d'espace complexe est plus générale que celle d'„espace analytique général" de H. Cartan [Séminaire H. Cartan de l'Ecole Normale Supérieure, 1951–52 et 1953–54; MR **16**, 233]. La définition d'une „modification" („modification continue" dans la terminologie des auteurs) peut être présentée comme suit: une application analytique $\tau: X' \to X$ de deux espaces complexes de même dimension est une modification s'il existe $N \subset X$, fermé et „mince" dans X, tel que, posant $N' = \tau^{-1}(N)$, la restriction de τ à $X' - N'$ soit un isomorphisme sur $X - N$; un sous-ensemble de X connexe est mince („dünn") s'il est contenu dans un vrai sous-ensemble analytique. Si $\tau: X' \to X$ est une modification, on montre qu'il existe un plus petit N jouissant de la propriété requise; $N' = \tau^{-1}(N)$ est alors l'ensemble des „points de dégénérescence" de τ, i.e. des x' tels que x' ne soit pas isolé dans $\tau^{-1}(\tau(x'))$; et $N = \tau(N') \cup (X - \tau(x'))$. Ces ensembles N et N' sont canoniquement définis par la modification τ.

Théorème: Si $\tau: X' \to X$ est une modification, et si X est une variété analytique complexe, de dimension (complexe) n, alors N' est un sous-ensemble analytique de dimension $n-1$ [résultat dû à Kreyszig quand X' est aussi une variété analytique, Math. Ann. **128**, 479–492 (1955); MR **16**, 689].

Une modification $\tau: X' \to X$ est „propre" si l'image réciproque de tout compact de X est un compact de X'. Alors $\tau(X') = X$, car $\tau(X')$ est fermé et dense dans X; $N = \tau(N')$ est un sous-ensemble analytique de X, d'après un résultat difficile de Remmert et Stein: dans toute application analytique propre, l'image d'un sous-ensemble analytique est un sous-ensemble analytique. Si $\tau: X' \to X$ est une modification propre, les composantes irréductibles de N sont de dimension $\leq n-2$ (avec $n = \dim X$), et l'homomorphisme $\tau^*: H(X) \to H(X')$ des anneaux de fonctions holomorphes est un isomorphisme (sur), parce que $H(X) \to H(X - N)$ est un isomorphisme et $H(X') \to H(X' - N')$ est un monomorphisme.

La notion de modification propre sert à définir une relation d'équivalence entre espaces complexes; deux espaces équivalents X et X' ont des anneaux $H(X)$ et $H(X')$ isomorphes.

Appelons „primitif" („Urraum") un espace complexe X' tel que toute modification $\tau: X' \to X$ soit un isomorphisme. Tout produit d'espaces projectifs complexes est primitif, en vertu du théorème: si $\tau: X' \to X$ est une modification propre, si X et X' sont paracompacts, et si K' est une composante irréductible de N', tout sous-ensemble analytique assez voisin de K' et de même dimension que K' est identique à K'.

Les auteurs étudient un exemple intéressant de modification $\tau: X' \to X$, où X' et X sont compacts (et même algébriques). Soit P l'espace projectif complexe de dimension 1; dans P^{2n+1}, décrit par $2n+1$ coordonnées complexes (finies ou infinies) t, w_i, z_i $(1 \leq i \leq n)$, soit X' défini par les équations $uw_i = vz_i$ (u et v désignant des coordonnées homogènes, finies, de $t \in P$); X' est une variété algébrique sans singularité, de dimension $n+1$. La projection $(t, w_i, z_i) \to (w_i, z_i)$ applique X' sur un sous-ensemble algébrique normal $X \subset P^{2n}$, sans singularité autre que l'origine 0. Alors $X' \to X$ est une modification, N' est isomorphe à P, N est réduit au point 0. Sur X, t est une fonction méromorphe ayant 0 pour unique point d'indétermination, quelle que soit la dimension $n+1$; chaque sous-ensemble analytique $t = \text{cte}$ de X ne peut pas être défini, au voisinage de 0, par moins de n équations holomorphes, bien que sa dimension soit $n = \dim(X) - 1$.

H. Cartan (Paris).

Grauert, Hans. Métrique kaehlérienne et domaines d'holomorphie. C. R. Acad. Sci. Paris **238**, 2048–2050 (1954).

Une métrique $d(p, q)$ sera dite complète dans une variété M à base dénombrable si pour toute suite infinie de points $p_i \in M$, sans point d'accumulation dans M on a $d(p_i, q) \to +\infty$, $q \in M$ fixe. De l'existence d'une métrique kählérienne complète dans un domaine D de l'espace C^n, on conclut à la pseudo-convexité de D dans certains cas: (1) pour qu'un domaine de Reinhardt admette une métrique kählérienne complète, il faut et il suffit qu'il soit domaine d'holomorphie ou résulte d'un tel domaine par suppression d'espace C^s coordonnées $[z_{i_1} = 0, \cdots, z_{i_s} = 0]$, $2 \leq s \leq n$; (2) pour qu'une métrique hermitienne dans M soit kählérienne, il faut et il suffit qu'une variété analytique complexe quelconque W^1 y soit une surface minima; en utilisant cette propriété, on montre qu'un domaine M de C^n ayant une frontière analytique réelle à $2n-1$ dimensions réelles et admettant une métrique kählérienne complète est pseudo-convexe, c'est à dire convexe par rapport aux fonctions plurisousharmoniques: un tel domaine est alors domaine d'holomorphie; ce dernier énoncé résulte (mis à part le cas des domaine de Hartogs dans C^2 qui se ramènent comme il est bien connu aux domaines de C^1) d'un théorème fondamental de K. Oka [Tôhoku Math. J. **49**, 15–52 (1942); MR **7**, 290] et de son extension récente au cas de $n > 2$ variables [voir le mémoire analysé ci-dessous].

P. Lelong (Lille).

Norguet, François. Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes. (Passage du local au global.) Bull. Soc. Math. France **82**, 137–159 (1954).

K. Oka [Tôhoku Math. J. **49**, 15–52 (1942), Th. 1; MR **7**, 290; we refer to this paper as Oka VI] has proved for two complex variables that every schlicht finite region which is locally a region of holomorphy is globally a region of holomorphy. (A region D in C^n (=space of n complex variables) is "locally a region of holomorphy" if for every point of C^n there exists a neighborhood V such that the intersection $D \cap V$ is a region of holomorphy or the empty set.) In proving Th. 1 Oka succeeded in solving a problem which had been open since 1911 [Behnke and Stein, Nieuw Arch. Wisk. (2) **23**, 227–242 (1951); MR **13**, 339]. Oka's proof essentially uses properties of C^2 and cannot automatically be extended to C^n. A proof for C^n is the subject of this paper. (Further proofs for $n \geq 2$: K. Oka IX, paper reviewed second below; H. J. Bremermann, paper reviewed below. Also a proof for $n = 2$, very close to Oka's proof, is given in B. A. Fuks, Uspehi Mat. Nauk **5**, no. 4 (38), 75–120 (1950), translated in Amer. Math. Soc. Transl. no. **93** (1953); MR **12**, 252; **15**, 116.) The papers of Norguet and Bremermann are similar in many points, both papers follow the basic ideas of Oka VI

and overcome the difficulties arising in the passage from C^2 to C^n in a similar way. The new proof of Oka IX, however, is completely different and valid not only for schlicht domains but for concrete complex manifolds over the C^n with no branch points as interior points.

The author shows first that Th. 1 is equivalent to the following Th. 1': Let x be the vector of the n complex variables $(x_1, \cdots, x_n)$, let $\Im x_j$ be the imaginary part of x_j, let $\mathfrak{D}$ be a bounded region and a_1 and a_2 two real numbers with $a_1<a_2$. If then $\{x|\Im x_1>a_1\}\cap\mathfrak{D}$ and $\{x|\Im x_1<a_2\}\cap\mathfrak{D}$ are regions of holomorphy, then $\mathfrak{D}$ is a region of holomorphy. For the proof of the equivalence of 1 and 1' the author introduces the following general notions: A class K of regions of the space R^m of m real variables $(\xi_1, \cdots, \xi_m)$ is called "intersection class" if it satisfies the following conditions: a) K contains the empty set and the "intervals" of R^m, that is the domains $\cap_{1\le i\le m}\{(\xi_1, \cdots, \xi_m)|a_i<\xi_i<b_i\}$; b) if $D_1 \in K$ and $D_2 \in K$, then $D_1\cup D_2 \in K$. The class K is called "closed to the right" if the limit of every increasing sequence of bounded regions of K belongs to K. Replace in theorems 1 and 1' "region of holomorphy" by "belongs to K". The author proves the equivalence of the thus obtained statements for general intersection classes which are closed to the right. The equivalence of theorems 1 and 1' for regions of holomorphy then follows from the fact that the class of holomorphy regions is an intersection class which is closed to the right. The latter is the content of the theorem of Behnke-Stein on convergent sequences of holomorphy regions. The author then generalizes the notion of holomorph-convexity (of Cartan-Thullen) to: A region is "holomorph-convex (holomorph-concave) on a part of its boundary". Using these and some more notions which he introduces, he derives "subtraction theorems" for regions of holomorphy which are helpful in later proofs. Then he transforms the integral formula which H. Cartan [Séminaire H. Cartan de l'Ecole Norm. Sup., 1951–52, exposé 6; MR 16, 233] has substituted for the Weil-Bergman formula into a form suitable for him. Thereafter he solves (somewhat roughly speaking) the following problem: Let G be a region of holomorphy, divided by the hyperplane $L=\{x|\Im x_1=0\}$ into G_1 and G_2. Given a function g in a neighborhood of $G\cap L$, find a pair of functions f_1, holomorphic in $\bar{G}_1\cap G$, and f_2, holomorphic in $\bar{G}_2\cap G$, such that $f_1-f_2=g$. To obtain the solution the Weil-Cartan integral is applied to g, split into two parts which define a pair of functions f_1 and f_2 which possess the required properties. A similar problem can be solved if instead of knowing that G is a region of holomorphy one knows only that $G\cap\{x|\Im x_1>a_1\}$ and $G\cap\{x|\Im x_1<a_2\}$, with $a_1<a_2$, are regions of holomorphy (and that G satisfies some additional conditions). To solve this problem the preceding solution for regions of holomorphy is applied to $G\cap\{x|a_1<\Im x_1<a_2\}$, then the meromorphic kernels of the integrals are approximated by functions which are meromorphic in the required regions, and finally the error which results from the approximation is compensated by solving an integral equation. Having solved this problem it follows that the additive Cousin problem can always be solved for a certain subregion D of G if the poles are given in a neighborhood of D. Using this result and the other properties of G and D (here not specified) it is concluded that D is a region of holomorphy. Finally the region $\mathfrak{D}$ of Th. 1' is approximated by regions G and D which possess all the properties required. Because of the theorem of Behnke-Stein it follows that $\mathfrak{D}$ is a region of holomorphy.

H. J. Bremermann (Princeton, N.J.).

Bremermann, Hans J. Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen. Math. Ann. **128**, 63–91 (1954).

E. E. Levi [Ann. Mat. Pura Appl. (3) **17**, 61–87 (1910)] showed that every domain of holomorphy with a sufficiently smooth boundary is a pseudoconvex domain, and he posed the converse problem: Is every pseudo-convex domain a domain of holomorphy? This problem remained unsolved for many years. For the case of two complex variables K. Oka [Tôhoku Math. J. **49**, 15–52 (1942); MR **7**, 290] answered the question as follows: A schlicht and finite domain is a domain of holomorphy if and only if it is a pseudo-convex domain. In the paper being reviewed the author obtains the same result for the case of n variables $n\ge 2$. [The author points out that after his paper was completed his attention was called to a hectographed manuscript by F. Norguet on the same problem and with the same answer. This paper by Norguet has now appeared and is reviewed above.] The concept of pseudo-convexity can be taken in several equivalent ways; the one used here is what is referred to as pseudo-convex in the sense of Cartan. A domain H is called pseudo-convex in this sense if about every boundary point P of H there is a hypersphere containing P whose intersection with H is a domain of holomorphy.

The main (and more difficult) part of the desired result is the part "if pseudo-convex, then domain of holomorphy", that is the part which goes from the local property to the property in the large. In a manner similar to Oka's work this part can be easily reduced to the following lemma: "Let D be a schlicht bounded region in the space of n complex variables. Let $a_1<a_2$ and assume that the regions $D_1=D\cap\{z|\,x_1>a_1\}$ and $D_2=D\cap\{z|\,x_1<a_2\}$ are regions of holomorphy. Then D is a region of holomorphy." Here z denotes the n variables $z_1=x_1+iy_1$, $\cdots$, $z_n=x_n+iy_n$. The main part of the paper is devoted to a proof of this lemma which is carried out by a series of approximations. In its proof the author makes important use of (1) Cousin's first problem; (2) the Bergman and Weil integral formulas; (3) the theorem of Behnke and Stein which states that the limit H of a sequence $\{H_\nu\}$ of regions of holomorphy is itself a region of holomorphy; (4) the theorem of Cartan and Thullen which states that if corresponding to every boundary point P of a region H there is a function which is holomorphic in H and singular at P, then H is a region of holomorphy; and (5) properties of the distance function (introduced by the author) $d_B(P)$=min dist. (P, Q), the minimum being taken with respect to all boundary points Q of B. The author uses (1), (2), (4) and (5) in order to obtain a sequence of regions interior to D and approaching D as a limit, each of which is a domain region of holomorphy. Then, using (3), he has the lemma. In order to economize on space the reviewer has given merely this brief resumé. A more detailed account of the proof would indicate how the limiting process (3) is used repeatedly; it would also show how the approximating subregions enter by use of (5) in such a way that (2) and a modification of (1) can be applied; and it would show how this modification of (1) enables the author to use (4). *W. T. Martin* (Cambridge, Mass.).

Oka, Kiyoshi. Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur. Jap. J. Math. **23** (1953), 97–155 (1954).

The author states that he was stimulated by the survey of H. Behnke and P. Thullen, "Theorie der Funktionen

mehrerer komplexer Veränderlichen" [Springer, Berlin, 1934], to solve the outstanding problems of the theory described in the survey. So far he has dedicated his life's work (which is represented by a series of papers in which the present one is the ninth) exclusively to this task. The problems are: The inverse problem of Hartogs (also known as the problem of E. E. Levi), the first and second Cousin problems, and the problem of approximating a holomorphic function by functions holomorphic in larger domains. For schlicht and finite domains in the space of n complex variables the author has solved these problems (which turn out to be intimately related) in parts I–VI [J. Sci. Hiroshima Univ. Ser. A. **6**, 245–255 (1936); **7**, 115–130 (1937); **9**, 7–19 (1939); Jap. J. Math. **17**, 517–521, 523–531 (1941); Tôhoku Math. J. **49**, 15–52 (1942); MR **3**, 85; **7**, 290]. From part VII on [Bull. Soc. Math. France **78**, 1–27 (1950); J. Math. Soc. Japan **3**, 204–214, 259–278 (1951); MR **12**, 18; **13**, 454; **14**, 264] he pursues the solutions for non-schlicht domains. The author has developed ingenious original methods which have had a profound influence on the theory of several complex variables.

The present paper generalizes the methods and results developed in the previous Papers I–VIII to "domaines finis sans point critique intérieur" (that is concrete complex manifolds over the space of n complex variables $(x)=(x_1, \cdots, x_n)$, where the $x_1, \cdots, x_n$ are local coordinates in a neighborhood of every point. "Domaines étalés" in the terminology of H. Cartan). Theorem I: Let $D \subset D'$ and let D be holomorph-convex with respect to functions holomorphic in D'. Then any function holomorphic in D can in any compact subdomain of D be uniformly approximated by functions holomorphic in D'. Theorem II: The first Cousin problem is always solvable for holomorph-convex domains. [Remark: Holomorph-convex domains of the type considered by the author are special "Stein manifolds" (in the terminology of H. Cartan). Theorem II was also obtained by H. Cartan for general Stein manifolds [Colloque sur les fonctions de plusieur variables, Bruxelles, 1953, Thone, Liège, 1953, pp. 41–55; MR **16**, 235]. Theorem III: Every pseudo-convex domain is holomorph-convex [a Stein manifold]. Corollary: Every pseudo-convex domain is a domain of holomorphy. (Solution of the inverse Hartogs problem.) [Remark: This theorem gives rise to a number of criteria for a concrete complex manifold to be a Stein manifold. It also solves affirmatively the problem whether the domains of holomorphy are holomorph-convex. (The holomorphy domains are pseudo-convex and hence by Theorem III holomorph-convex.) This problem has been open for non-schlicht domains for many years. (Compare H. Cartan, loc. cit., p. 50, No. 4.) Theorem III solves also another problem stated by H. Cartan [loc. cit., p. 54] for "domaines étalés": Let X be a concrete complex manifold without branch points as interior points. Then X is a Stein manifold if (and only if) it is locally a Stein manifold.] Theorem IV: The second Cousin problem has a holomorphic solution if it has a continuous solution. [Cf. also J. P. Serre, ibid., pp. 57–68; MR **16**, 235.]

The inverse Hartogs problem occupies by far the most space in the present paper and utilizes the solutions of all the other problems. [Solutions for schlicht domains in n variables were also given by F. Norguet in the paper reviewed second above and by H. J. Bremermann in the paper reviewed above.] In his paper VI (which solves the inverse Hartogs problem for schlicht domains in two variables) the author uses essentially the Weil-Bergman integral formula (and so do Norguet and Bremermann). This formula is not available for non-schlicht domains. Therefore the author replaces the Weil-Bergman formula by an imbedding of analytic polyhedra as analytic surfaces into higher-dimensional polycylinders, continuation of the functions from the surface into the polycylinder and then applying the classical Cousin integral together with other methods of his papers I and II.

Another important tool in the paper are the pseudo-convex domains and functions. The author defines three "Kontinuitätssätze A, B, C", and defines a domain to be "pseudo-convex A, B, C" if it satisfies Kontinuitätssatz A, B, or C respectively, and if this property is invariant with respect to pseudo-conformal transformations. He then shows that the three pseudo-convexities are equivalent. [Remark: They are also equivalent with the pseudo-convexity defined by P. Lelong and H. J. Bremermann.] The "pseudo-convex functions" were introduced by the author in his paper VI and independently and at about the same time under the name "plurisubharmonic functions" by P. Lelong [C. R. Acad. Sci. Paris **215**, 398–400, 454–456 (1942); **216**, 107–109 (1943); MR **5**, 123, 124, 177]. A function is "pseudo-convex" (plurisubharmonic) if 1) it is upper semicontinuous and 2) its restriction to analytic planes of one complex dimension is subharmonic. (Lelong excludes the constant $-\infty$.) The connection between pseudo-convex domains and functions is: In a pseudo-convex domain D the function $-\log d(P)$, where $d(P)$ is the euclidean distance of the point P from the boundary of D, is a pseudo-convex function in D. [Remark: Several of the results on pseudo-convex functions in this paper have also been obtained by P. Lelong, Ann. Sci. École Norm. Sup. (3) **62**, 301–338 (1945); J. Analyse Math. **2**, 178–208 (1952); MR **8**, 271; **14**, 971; and H. J. Bremermann, Schr. Math. Inst. Univ. Münster no. **5** (1951); MR **14**, 971.] Important is the following result of the author (for non-schlicht domains!): Let D be a pseudo-convex domain. Then one can find a function $\Phi(P)$, pseudo-convex in D, such that for every real α:

$$\{P \mid P \in D \wedge \Phi(P) < \alpha\} \subset\subset D.$$

Finally the author extends solutions of several "problèmes arithmétiques" to non-schlicht domains and gives a counterexample for his "problem E".

H. J. Bremermann (Princeton, N.J.).

Hitotumatu, Sin. On some conjectures concerning pseudo-convex domains. J. Math. Soc. Japan **6**, 177–195 (1954).

The author summarises some main results from the theory of plurisubharmonic functions and pseudo-convex domains, the conjecture of Bochner and Martin that every plurisubharmonic function is a Hartogs function, and some consequences of this conjecture. He presents two new necessary and sufficient conditions for a domain to be pseudo-convex. (Recently the reviewer, in an unpublished paper, has disproved the conjecture of Bochner-Martin and the (consequence)-conjecture that any plurisubharmonic function has a plurisubharmonic continuation into the envelope of holomorphy of its domain of definition. The disproof of the conjecture provides also a counterexample to proposition 18, p. 188, which is originally due to P. Lelong, [Ann. Sci. École Norm. Sup. (3) **62**, 301–338 (1945), theorem 10, p. 331; MR **8**, 271]. Lelong overlooks in his proof that a locally supporting plane of a convex surface is in general not a globally supporting plane if the projection of the surface is not a

convex domain.) Then the author gives some equivalences for the conjecture that every pseudo-convex domain is a domain of holomorphy. Meanwhile this conjecture has been proved (as the author remarks) by K. Oka, paper reviewed above, F. Norguet, paper reviewed third above, and the reviewer, paper reviewed second above.

H. J. Bremermann (Princeton, N.J.)

Rothstein, Wolfgang. Zur Theorie der analytischen Mannigfaltigkeiten im Raume von n komplexen Veränderlichen. Math. Ann. **129**, 96–138 (1955).

The author continues his previous investigations on the continuation and singularities of analytic sets in the space of n complex variables C^n [Math. Ann. **121**, 340–355 (1950); **122**, 424–434 (1951); **126**, 221–238 (1953); MR **11**, 652; **12**, 818; **15**, 616]. An analytic set is a set in a domain of the C^n which is locally the simultaneous zero manifold of a finite number of holomorphic functions. The continuation of analytic sets and the singularities show certain analogies with continuation and singularities of holomorphic functions.

The main theorem is the following (analog to the "theorem of Hartogs"): Let the complex variables be $z=(z_1, \cdots, z_p)$ and $w=(w_1, \cdots, w_q)$. Let $Z(r)$ be the polycylinder $\{z \mid |z_1|<r, \cdots, |z_p|<r\}$ and

$$W(r)=\{w \mid |w_1|<r, \cdots, |w_q|<r\}.$$

Let g^k be a complex k-dimensional irreducible analytic set in the domain $\{Z(\varepsilon), W(1)\}\cup\{Z(1), W(1)-W(\frac{1}{2})\}$ of the $C^n=C^{p+q}$. Let $k\geq 2$ and $q+k\geq n+1$. Let g^k approach arbitrarily the boundary surface $\{Z(1)$, boundary of $W(1)\}$. Then there exists an irreducible k-dimensional analytic set g_*^k in $\{Z(1), W(1)\}$ which contains g^k and is such that $g_*^k=g^k$ in $\{Z(1), W(1)-W(\frac{1}{2})\}$.

The local continuation of analytic sets into sets $\{z|\varphi(z)<0\}$ is also studied, where $\{z|\varphi(z)=0\}$ is a hypersurface. In this connection the author refines the Levi-Krzoska condition for $\{z|\varphi(z)=0\}$ to be pseudo-convex:

$$\sum_{\mu,\nu=1}^{n}\frac{\partial^2\varphi}{\partial z_\mu \partial\bar{z}_\nu}dz_\mu d\bar{z}_\nu\geq 0 \text{ for } \sum_{\mu=1}^{n}\frac{\partial\varphi}{\partial z_\mu}dz_\mu.$$

If the index of the hermitean form is q at a point ζ, then a q-dimensional analytic surface passes through ζ not intersecting $\{\varphi<0\}$ in a neighborhood of ζ.

H. J. Bremermann (Princeton, N.J.).

Hirzebruch, Friedrich. Übertragung einiger Sätze aus der Theorie der algebraischen Flächen auf komplexe Mannigfaltigkeiten von zwei komplexen Dimensionen. J. Reine Angew. Math. **191**, 110–124 (1953).

Soit M une variété analytique complexe compacte, de dimension complexe deux, et soit H un sous-ensemble analytique principal de M. Théorème 1: Si H est sans singularité, soient $e(H)$ sa caractéristique d'Euler-Poincaré, h la classe d'homologie de H et c_1 la classe d'homologie duale de la 2-classe de Chern de M; alors, si $a\circ b$ désigne le nombre d'intersection des classes d'homologie a et b, on a: $e(H)=-h\circ h+c_1\circ h$. Si H a des singularités, on applique à M le processus σ de H. Hopf [voir Hirzebruch, Math. Ann. **126**, 1–22 (1953); MR **16**, 26], en chacun des points singuliers de H, de manière que la transformée H^* de H soit une sousvariété (sans singularité) de la transformée de M. Alors, en posant $e(H)=e(H^*)$, la formule précédente est valable après adjonction, au second membre, d'entiers positifs dépendant seulement de la structure locale des singularités. Si M est une surface algébrique, en remarquant que $-c_1$ est la classe canonique de M, on retrouve la formule donnant le genre de la courbe H. Viennent ensuite des théorèmes généralisant des résultats classiques sur les faisceaux de courbes algébriques d'une surface algébrique; en particulier: Théorème 2: Soit F une fonction méromorphe sur M. Les éléments de surface tangents aux surfaces de niveau de F (dimension réelle 2) constituent un champ à nombre fini de singularités, dont la somme des indices [voir H. Hopf, Colloq. Internat. Centre Nat. Rech. Sci., no. 12, Paris, 1949, pp. 55–59; MR **11**, 610] est

$$(2h-w)\circ(2h-w)-(2h-w)\circ c_1+c_2,$$

où h est la classe d'homologie commune aux surfaces de niveau de F, w une classe d'homologie déterminée par F (ramification de F) et c_2 la caractéristique d'Euler-Poincaré de M. La technique des démonstrations repose sur des méthodes de H. Hopf [loc. cit.] et de E. G. Kundert [Ann. of Math. (2) **54**, 215–246 (1951); MR **13**, 374].

P. Dolbeault (Paris).

Gunning, R. C. General factors of automorphy. Proc. Nat. Acad. Sci. U.S.A. **41**, 496–498 (1955).

D étant une variété simplement connexe à structure analytique complexe, à groupe d'automorphismes analytiques Γ, on suppose D/Γ compact, l'ensemble des Γz sans point limite dans D, et l'existence de voisinages tels que $TU\cap U\neq 0$ pour tout $T\in\Gamma$ entraine $TU=U$; de plus, pour chaque couple z_1, z_2 tel que $\Gamma z\neq\Gamma z_2$ il existe U_1 et U_2 tels que $\Gamma U_1\cap\Gamma U_2$ est vide; enfin il existe une métrique kaehlerienne sur D invariante par Γ. On étudie alors les facteurs d'holomorphie de Γ sur D [cf. S. Bochner, J. Indian Math. Soc. (N.S.) **11**, 1–21 (1947); MR **9**, 423], c'est à dire les ensembles de fonctions $\nu_T(z)$ holomorphes sur D qui vérifient

$$\nu_{ST}(z)=\nu_S(Tz)\nu_T(z)$$

et l'on définit des classes de caractères. Deux ensembles $\{\nu_T(z)\}$, $\{\mu_T(z)\}$ sont alors équivalents si et seulement si ils ont la même classe de caractères; l'équivalence a été préalablement définie par l'existence de constantes c_T et d'une fonction $h(z)$ holomorphe sans zéro dans D telle qu'on ait

$$\mu_T(z)=c_T\nu_T(z)h(Tz)/h(z).$$

Etude de cas particuliers [notamment D borné, les formes (1, 1) fermées invariantes par Γ ayant comme périodes sur certains 2-cycles des multiples de la forme de courbure de Ricci, hypothèse qui permet de généraliser des propriétés étudiées pour une seule variable complexe [H. Petersson, Math. Ann. **115**, 23–67 (1937)]. Extension à des facteurs représentés par des matrices. *P. Lelong.*

Kawada, Yukiyosi. On analytic line bundles with the affine structural group. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **4**, 85–91 (1954).

In this note the author considers fiber bundles over a compact complex analytic manifold, V, with fiber the complex numbers and structural group, G, the one-dimensional affine group. After Weil, he introduces for every such affine bundle a subordinate complex line bundle, A. There are several affine bundles to which A is subordinate; denote these by $B(A)$. The author considers $H^1(V, S(A))$, where $S(A)$ is the sheaf of germs of local holomorphic cross-sections of A. Each B in $B(A)$ determines a cohomology class $c(B)$ in $H^1(V, S(A))$. The affine bundle is reducible to A if and only if $c(B)$ is zero.

Next, the author compares the invariant $c(B)$ with a

similar invariant introduced by Nakano [Proc. Japan Acad. **30**, 542–547 (1954); MR **16**, 737] and shows that the two are equivalent. For a non-singular algebraic variety the author shows that every affine bundle is defined by a system (U_j, a_{jk}, b_{jk}) where $a_{jk}(z)=d_j(z)/d_k(z)$ and $b_{jk}(z)=d_j(z)(s_j(z)-s_k(z))$ on $U_j\cap U_k$ and $d_j(z)$, $s_j(z)$ are meromorphic on U_j, which is the analogue of the known result for line bundles. With this, the writer shows that the condition $c(B)=0$ for a given affine bundle, B, is equivalent to the existence of a global meromorphic function $F(z)$ such that $d_j(z)(F(z)-s_j(z))$ is holomorphic on U_j.

Finally, the condition $c(B)=0$ is expressed in terms of residues and several formulas are derived, which in the case of algebraic curves express the inhomogeneous Riemann-Roch theorem. *P. E. Conner.*

Guggenheimer, Heinrich. Sopra una successione esatta e sulle modificazioni delle varietà kähleriane di dimensione quattro. Boll. Un. Mat. Ital. (3) **10**, 153–160 (1955).

On a Kähler manifold V introduce the exterior derivatives d', d'', and their adjoints $\delta'=-*d''*$, $\delta''=-*d''*$. The author considers, in addition to the d'- and d''-cohomology groups of differential forms with compact support, the groups

$$\{\varphi|d'\varphi=d''\varphi=0\}/\{\varphi|\varphi=d'd''\psi\},$$

and analogous groups involving the pairs of operators δ', δ'' and d'', δ', on differential forms with compact support. All of these groups are isomorphic to the ordinary cohomology of V with compact support. Let $W\subset V$ be a Kähler submanifold; a Kähler modification of V at W is a transformation of the pair (V, W) into another such pair (V', W'), which is an analytic homeomorphism of $V-W$ onto $V'-W'$. By examining the cohomology sequences of the pairs (V, W), (V', W') associated to each of the five cohomology groups of differential forms introduced above, the author derives relations among the ranks of the cohomology groups. These relations are examined in particular when V is of complex dimension 4, and in that case lead to results of the following types: every subvariety W' (with the above notation) has geometric genus zero; every Kähler modification at a point transforms the point into a hypersurface.

R. C. Gunning (Chicago, Ill.).

Ishihara, Shigeru. Groups of isometries of pseudo-Hermitian spaces. I. Proc. Japan Acad. **30**, 940–945 (1954).

Let M be an almost Hermitian manifold of real dimension $2n$, G a group of Hermitian isometries of M. It is shown that G is transitive on M if $\dim G\geq n^2+2$ and $n\geq 2$. If $n\geq 3$ and $n\neq 4$, then $n^2+2n-1>\dim G>n^2+2$ (trivially $n^2+2n\geq\dim G$). The author then makes the further assumption that M is a homogeneous space, $M=G/H$, and investigates the cases: A) $\dim G=n^2+2n$, B) $\dim G=n^2+2n-1$. In case A) he finds M is Kählerian with constant holomorphic sectional curvature K. If $K>0$ and M is simply connected, then $G=\mathrm{SU}(n+1)$ and M=complex projective n-space; he also determines the cases $K<0$ and $K=0$. He also proves in case A) that, for $n\neq 4$, M factors locally into a two (real) dimensional and a $2(n-1)$-dimensional space, the factors being homogeneous and almost Kählerian. In case B) he proves M is flat and determines G.

W. Ambrose (Cambridge, Mass.).

Schouten, J. A., and Yano, K. On invariant subspaces in the almost complex X_{2n}. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17**, 261–269 (1955).

In a manifold X_{2n} let F_i^j (satisfying $F_i^jF_j^k=\delta_i^k$) represent an almost-complex structure. A submanifold X_{2m} is called invariant if for every tangent vector v^i at every point of X_{2m} also $F_j^iv^j$ is a tangent vector. The X_{2m} has then an induced almost-complex structure (Theorem 1), and if X_{2n} is pseudo-complex (i.e. the torsion N_{ij}^k vanishes), then X_{2m} is also pseudo-complex (Theorem 2). An almost (pseudo)-Hermitian metric in X_{2n} induces the same in X_{2m} (Theorem 3), and an almost (pseudo)-Kählerian structure behaves likewise (Theorem 4). The intrinsic connection [Schouten and Yano, same Proc. **58**, 1–9 (1955); MR **16**, 858.] in X_{2m} derived from an induced almost-Hermitian metric is the same as the connection induced in X_{2m} by the intrinsic connection in X_{2n} and the imbedding (Theorem 6).

A remarkable result is that every invariant X_{2m} in an almost-Kählerian X_{2n} is always minimal (Theorem 7). Another result is that an almost-Kählerian manifold $\partial_ig^{ij}F^{ij}=0$; i.e. for $\omega=F_{ij}dx^i\wedge dx^j$ one has $\delta\omega=0$ (Theorem 8). *A. Nijenhuis* (Chicago, Ill.).

Algebraic Geometry

Chisini, Oscar. Aspetti significativi della geometria algebrica. Confer. Sem. Mat. Univ. Bari no. **6**, 22 pp. (1955).

Expository.

Spampinato, Nicolò. Rappresentazione in S_5 di un fascio di curve e della serie lineare secata su una curva complessa prolungata nel campo biduale. Ricerca, Napoli **6**, no. 1, 18–25 (1955).

Turri, Tullio. Le trasformazioni involutorie dello spazio date dalle intersezioni di superficie di ordine quattro. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 165–176 (1955).

Turri, Tullio. Le trasformazioni birazionali involutorie dello spazio. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 256–265 (1955).

Masotti Biggiogero, Giuseppina. Sui rami ciclici di curve piane. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)**, 387–399 (1954).

The main theorem is: A necessary and sufficient condition that an algebraic branch with cartesian equation

$$y=ax^{(r+v)/r}+bx^{(r+v+v')/r}+cx^{(r+v+v'+v'')/r}+\cdots$$

(where r and the h.c.f's of (r, v), (r, v, v'), $\cdots$ are a strictly descending sequence) shall be transformed into itself by a homography of the form

$$x'=\lambda x,\quad y'=\mu y$$

is that v', v'', $\cdots$ all have a common divisor h, which is not a divisor of both r and v. The homography is then of period h. In particular, if h is a divisor of r, $r+v$, or v, then $\lambda=1$, $\mu=1$, or $\lambda=\mu$ respectively. Some applications are given, in particular to the unloading which may occur on the hessian of a curve with a tacnode.

P. Du Val (London).

Chisini, Oscar. Il teorema di esistenza delle trecce algebriche. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 143–149 (1954); **17** (1954), 307–311 (1955).

Démonstration de l'existence et éventuellement de l'irréductibilité d'une courbe algébrique, dont la tresse est déduite de la tresse canonique par modification de l'ordre des traits, et fusion de ceux-ci. La marche des démonstrations s'inspire des démonstrations géométriques du théorème de Riemann, prouvant l'existence d'une fonction algébrique, construite à partir d'une droite multiple dont sont donnés les points de diramation et le groupe de monodromie. *B. d'Orgeval* (Dijon).

Marchionna Tibiletti, Cesarina. Trecce algebriche di curve di diramazione: costruzioni ed applicazioni. Rend. Sem. Mat. Univ. Padova **24**, 183–214 (1955).

Construction des tresses des courbes de diramation φ de surfaces générales d'ordre n, à partir de la forme canonique de la tresse d'une courbe d'ordre $2n$ (cf. p. 191). L'A. donne également la tresse de la courbe de diramation d'une surface d'ordre m, dotée d'un point multiple isolé d'ordre h. Application du théorème de Chisini [voir l'analyse ci-dessus] pour obtenir de nouvelles courbes de diramation ψ à partir des précédentes. Démonstration de l'unicité du plan multiple obtenu à l'aide d'une courbe de diramation du type φ. Dans le cas des courbes ψ, existence du plan multiple mais il n'y a pas toujours unicité. Application au problème des surfaces avec un nombre élevé de points doubles isolés correspondants sur la courbe de diramation à des points doubles inessentiels. Exemple de la quintique de Togliatti avec 31 points doubles. *B. d'Orgeval* (Dijon).

d'Orgeval, B. Plans doubles non rationnels de genres $p_a=p_g=0$ à courbes de diramation du douzième ordre. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 23–33 (1955).

Tous les plans doubles non rationnels et de genre $p_a=p_g=0$, dont la courbe de diramation est d'ordre inférieur ou égal à 10, sont connus d'après les travaux de L. Campedelli. Il y en a dont la courbe de diramation est d'ordre 12. Dans ce cas les courbes canoniques du plan double devraient être d'ordre 3; et comme $p_a=p_g=0$ ces courbes n'existent pas; il s'ensuit que les multiplicités de la courbe de diramation doivent imposer aux courbes du troisième ordre plus que g conditions indépendantes. En considérant tous les cas possibles, compte tenu du fait que le courbe de diramation, d'ordre 12, ne doit contenir aucune composante multiple et ne saurait être formée avec les courbes d'une faisceau, on trouve qu'il y a six cas à discuter. La discussion est plutôt minutieuse et conduit à la conclusion qu'il y a des plans doubles du type désiré qui sont réalisables et qui appartiennent tous à des familles dont on connaissait déjà des représentants; d'autres cas conduisent à des plans doubles dont l'existence résulte ici seulement probable, la conclusion étant fondée seulement sur un calcul de paramètres libres. *E. G. Togliatti* (Gênes).

★ **Burniat, Pol. Sur l'existence de certaines surfaces canoniques multiples.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 107–109. Fédération belge des Sociétés Scientifiques, Bruxelles.

Les deux surfaces d'équations (en coordonnées homogènes x_0, x_1, x_2, x_3):

$$x_0^4x_3^{4s}+\lambda x_0^2x_3^{2s}\varphi_{2s}\varphi_2+\mu x_0x_3^s\varphi_{2s}^3+\varphi_{2s}^2\varphi_4=0,$$

où $s=2, 3$, où $\varphi_i=0$ est un cône de sommet $X_0(1, 0, 0, 0)$, le cône $\varphi_{2s}=0$ étant décomposé en $2s$ plans distincts passant par la droite $x_1=x_2=0$, ont le système canonique composé avec une involution I_4 d'ordre 4, généralement simple, qui est découpé sur la surface par les droites de la gerbe de centre X_0; les projections des courbes canoniques du point X_0 sur le plan $x_0=0$ sont des courbes d'ordre s passant $s-1$ fois par le point (0, 0, 1, 0). On a $p_a=p_g=2s+1$ et $p^{(1)}=8s-3$. Les singularités de ces deux surfaces sont assez compliquées par la présence de droites multiples voisines. *E. G. Togliatti* (Gênes).

Barker, C. C. H. Contact of surfaces on an algebraic fourfold. J. London Math. Soc. **30**, 343–350 (1955).

Let C be a non-singular irreducible algebraic curve on a non-singular irreducible algebraic V_4. If S and T are two algebraic surfaces of V_4 passing simply through C and having a k-contact along C ($k\geq 2$, $k=2$ corresponding to ordinary contact), then in general S and T have on C a certain number of binodes of order k constituting two sets of points, σ and τ respectively, and meet outside C in a set of points γ.

It is proved that, on C, the following equivalences hold:

$$(C^2)_S+\sigma\equiv(C^2)_T+\tau\equiv\delta \text{ say,}$$
$$\gamma\equiv(ST)_{V_4}-k\chi+k(k+1)\delta-k(\sigma+\tau),$$

where χ denotes the difference between a canonical set of C and the virtual intersection of C with a canonical primal of V_4. Also the isolated contact of two surfaces of V_4 at a simple point is studied, and the special case when V_4 is a linear space is considered. *B. Segre* (Rome).

★ **Derwidué, L. Sur la réduction des singularités d'une variété algébrique.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 100–103. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author sums up the arguments he has previously used in trying to prove that every algebraic variety can be transformed birationally into a variety which is free from singularities [Math. Ann. **123**, 302–330 (1951); MR **13**, 67]. However, those arguments cannot be taken as conclusive, as has been already pointed out by O. Zariski [MR 13, 67–70] and B. Segre [Ann. Mat. Pura Appl. (4) **33**, 5–48 (1952); MR **14**, 683]. The same author has returned later to the subject, making some improvements, but leaving still a number of important objectionable points [Mém. Soc. Roy. Liège (4) **13**, no. 1–2, 1–41 (1953); MR **15**, 551]. *B. Segre* (Rome).

★ **Defrise, Pierre. Variétés algébriques et idéaux.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 87–90. Fédération belge des Sociétés Scientifiques, Bruxelles.

Esquisse d'une méthode permettant de définir une notion „fine" de variété algébrique (c'est à dire avec points infiniment voisins et multiplicités): une telle „variété" est essentiellement un système d'idéaux à la Prüfer. Le cas des idéaux complets de Zariski est le plus important. Définition et étude des combinaisons linéaires formelles de telles variétés. Lien avec la théorie des valuations. *P. Samuel* (Clermont-Ferrand).

Northcott, D. G. The neighbourhoods of a local ring. J. London Math. Soc. **30**, 360–375 (1955).

Soient Q un anneau d'intégrité local et $\mathfrak{m}$ son idéal maximal. Un élément b de $\mathfrak{m}^s$ tel que $\mathfrak{m}^{s+n}:(b)=\mathfrak{m}^n$ pour tout n assez grand est dit superficiel de degré s. L'ensem-

ble R des éléments a/b pour lesquels il existe s tel que $a \in \mathfrak{m}^s$ et que b soit superficiel de degré s, est un anneau contenant Q. Les éléments b de $\mathfrak{m}^s$ qui sont superficiels de degré s sont ceux tels que $Rb=R\mathfrak{m}^s$. Au moyen d'une définition équivalente de R utilisant l'anneau gradué $\sum_s \mathfrak{m}^s/\mathfrak{m}^{s+1}$ associé à Q, on montre que R est noethérien. Lorsque Q contient un élément v superficiel de degré 1 (ce qui est le cas lorsque $Q/\mathfrak{m}$ est un corps infini), R est un anneau de fractions d'un anneau de la forme

$$Q[u_1/v, \cdots, u_r/v] \quad (u_i \in \mathfrak{m}).$$

Etude des anneaux locaux principaux du premier voisinage de Q, c'est à dire des anneaux $Q_i=R_{\mathfrak{p}_i}$ où les $\mathfrak{p}_i$ sont ceux des idéaux premiers associés à $R\mathfrak{m}$ qui sont de même dimension que Q: ils sont de dimension 1, et on a l'inégalité $e(Q) \geq \sum_i e(Q_i)$ entre les multiplicités des anneaux locaux Q et Q_i. Etude du cas où Q est de dimension 1; dans ce cas Q est régulier si et seulement si tous les Q_i lui sont identiques. Par application de la même construction aux anneaux locaux Q_i, on obtient les anneaux locaux Q_{ij} du second voisinage de Q, et ainsi de suite; ceci donne un arbre d'anneaux locaux, qui se ramifie seulement en un nombre fini de points. Dans le cas des anneaux locaux géométriques, cet arbre contient seulement un nombre fini d'anneaux distincts. *P. Samuel* (Clermont-Ferrand).

Northcott, D. G. A note on the genus formula for plane curves. J. London Math. Soc. **30**, 376–382 (1955).

En utilisant les notions introduites dans le précédent article, l'auteur donne une interprétation algébrique des points multiples infiniment voisins d'une courbe plane C, et montre comment calculer le genre g de C au moyen de son degré n et des anneaux locaux de ses points singuliers. Etant donné un anneau local Q, on note $e(Q)$ sa multiplicité et $d(Q)$ l'entier $e(Q)(e(Q)-1)$. Soit d la somme $\sum d(Q)$ étendue à tous les anneaux locaux des points de C et à tous les anneaux locaux de leurs arbres (on a $d(Q)=0$ lorsque Q est régulier, et cette somme a donc seulement un nombre fini de termes non nuls). On montre alors, au moyen de transformations quadratiques, que l'on a $2g=(n-1)(n-2)-d$. *P. Samuel* (Clermont-Ferrand).

Lluis, Emilio. Sur l'immersion des variétés algébriques. Ann. of Math. (2) **62**, 120–127 (1955).

L'auteur démontre que toute variété algébrique projective non singulière de dimension r est en correspondance birégulière (par projection) avec une variété de l'espace projectif de dimension $2r+1$, résultat connu dans le cas classique, et démontré ici algébriquement en caractéristique quelconque. Ce mémoire contient aussi une généralisation au cas des variétés avec singularités, et divers résultats sur les sécantes limites et les cones de tangentes. *P. Samuel* (Clermont-Ferrand).

Igusa, Jun-ichi. A fundamental inequality in the theory of Picard varieties. Proc. Nat. Acad. Sci. U.S.A. **41**, 317–320 (1955).

Soit f une application rationnelle d'une variété V dans une variété U. Si f est séparable, c'est à dire si $k(M)$ est séparable sur $k(f(M))$ (M: point générique de V sur k) f définit un homomorphisme δf de l'espace des formes différentielles sur U qui sont régulières en $f(V)$ dans l'espace des formes différentielles sur V. Propriétés fonctorielles de δf. Si U est non singulière, δf applique l'espace $E(U)$ des formes différentielles de première espèce et de degré 1 sur U dans $E(V)$; nous dirons que δf a la propriété d'isomorphisme si la restriction de δf à $E(U)$ est biunivoque. Soit alors V une surface non singulière, (C_u) un faisceau linéaire convenablement choisi de courbes sur V, et J le lieu des jacobiennes des C_u; il existe une application birationnelle g de V dans J (construite au moyen des fonctions canoniques des C_u) telle que δg ait la propriété d'isomorphisme; par composition on en déduit que, si f est l'application canonique de V dans sa variété d'Albanese A, δf a aussi la propriété d'isomorphisme. Généralisation au cas d'une surface quelconque par désingularisation [cf. Zariski, Ann. of Math. (2) **40**, 639–689 (1939); **43**, 583–593 (1942); MR **1**, 26; **4**, 52; Abhyankar (sous presse)], puis au cas d'une variété quelconque par section hyperplane générique. Comme A est une variété abélienne, la dimension de $E(A)$ est égale à la dimension q de A (c.à.d. à l'irrégularité de V); d'où l'inégalité $q \leq \dim(E(V))$. L'égalité est vraie en caractéristique 0, mais fausse en caractéristique p (contre-exemple, non publié, dû à J. Igusa). *P. Samuel* (Clermont-Ferrand).

Igusa, Jun-ichi. Arithmetic genera of normal varieties in an algebraic family. Proc. Nat. Acad. Sci. U.S.A. **41**, 34–37 (1955).

Soient V et V' deux variétés projectives telles que V' soit spécialisation de V, et que V' soit normale. Alors V est normale. Si D et D' sont des diviseurs sur V et V' tels que $(V, D) \to (V', D')$, et si D est linéairement équivalent à 0 sur V, alors D' est linéairement équivalent à 0 sur V' (méthode classique des sections, et utilisation du fait que, si une courbe non singulière C' est spécialisation d'une courbe C, alors C est non singulière et a même genre g que C'; ce dernier résultat est démontré de façon fort simple en remarquant que $2-2g$ est le nombre d'intersections de la diagonale de $C \times C$ avec elle même). D'autre part V et V' ont même polynome caractéristique de Hilbert donc même genre arithmétique; ainsi le genre arithmétique est constant pour les variétés normales d'un même système irréductible. *P. Samuel* (Clermont-Ferrand).

Lang, Serge. Abelian varieties over finite fields. Proc. Nat. Acad. Sci. U.S.A. **41**, 174–176 (1955).

Soient K_1/k_1 un corps de fonctions algébriques, et k un sous corps de k_1. L'auteur donne une condition suffisante pour que K_1 provienne, par extension du corps de base, d'un corps de fonctions algébriques K/k. Si K/k est un corps de fonctions algébriques, et si k_1 est une extension de k telle que Kk_1/k_1 soit un corps de fonctions abéliennes, il existe un modèle V de K/k qui est birégulièrement équivalent à une variété abélienne sur k_1. De plus, si k_1 est séparable sur k, V est une variété abélienne sur k si et seulement si elle admet un point rationnel sur k. Supposons maintenant que k soit un corp fini à q éléments, et que V soit une variété définie sur k et admette une structure de variété abélienne sur la clôture algébrique de k; l'étude de l'application $x \to x^q$ de V sur V montre que V admet un point rationnel sur k, et est donc une variété abélienne sur k. *P. Samuel* (Clermont-Ferrand).

Mattuck, Arthur. Abelian varieties over p-adic ground fields. Ann. of Math. (2) **62**, 92–119 (1955).

Let k be an ultrametric field (i.e. a field with a real-valued non-archimedian valuation) which is complete and of characteristic 0, and let A be a d-dimensional Abelian variety over k. The author proves that the group of points of A rational over k contains a subgroup isomorphic to the direct sum of d times the additive group of integers of k; moreover, if k is locally compact, then this subgroup is of finite index (which implies that A

has only a finite number of points of finite orders rational over k). The theorem is first proved in the case where A is the Jacobian variety of a curve C rational over k and having a rational point over k. In this case, the method is an adaptation of the proof that J is a torus group in the case where the basic field is that of complex numbers. Denoting by $\mathfrak{m}_1+\cdots+\mathfrak{m}_g=\mathfrak{M}$ (g=genus of C) a suitable non-special divisor of degree g of C, the points of a neighbourhood of 0 in J are represented by the divisors $\mathfrak{P}-\mathfrak{M}$, where $\mathfrak{P}=\mathfrak{p}_1+\cdots+\mathfrak{p}_g$, each $\mathfrak{p}_i$ being a point near $\mathfrak{m}_i$ on C. Any differential du of the first kind on C defines by integration a function u_i on a neighbourhood of $\mathfrak{m}_i$, and $\sum_{i=1}^{g} u_i(\mathfrak{p}_i)=v(\mathfrak{P}-\mathfrak{M})$ defines a holomorphic function v on on a neighbourhood of 0 in J. If $du_1, \cdots, du_g$ form a base of the space of differentials of the first kind, the corresponding functions v yield a uniformization of a neighbourhood of 0 in J, and this uniformization transforms the addition in J into the addition in a g-dimensional vector space over k; the theorem now follows easily. To go over to the general case, the author imbeds the Abelian variety A into the Jacobian variety of a curve with a rational point over k; this is possible because A, having a rational point over k (the zero element), has, in the neighbourhood of this point, points which are rational over k but generic over a suitable field of definition of A, from which it follows that A contains sufficently general curves having rational points over k. One then uniformizes locally the ambient Jacobian variety J in the manner indicated above; since A is a subgroup of J, it will be represented locally by a linear variety in the parameter space, from which the theorem follows. *C. Chevalley* (Paris).

NUMERICAL ANALYSIS

Cohn, Harvey. A numerical study of quintics of small discriminant. Comm. Pure Appl. Math. **8**, 377–385 (1955).

A study of all irreducible quintics

$$x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5,$$

in which the a's are integers not exceeding 3 in absolute value, was made to find those having small discriminants. The work was done on the UNIVAC. Six quintics like

$$x^5-x^3+x^2+x-1 \text{ and } x^5-3x^3+2x-1$$

have a discriminant of 1609 which the author believes is the minimum possible discriminant. A second investigation allowed the a's to be as large as 6 in absolute value but was restricted to the totally real case and was somewhat hampered by overflow.

Here the contest was won by Sylvester's cyclotomic quintic

$$x^5+x^4-4x^3-3x^2+3x+1$$

with discriminant $11^4=14641$. The author lists 76 quintics having small discriminants. All but one of these have $|a|_5=1$. In the first investigation "small" is defined as lying between -8000 and 3000; in the second it is less than 50000. *D. H. Lehmer* (Berkeley, Calif.).

Markovitch, Dragoljub. Sur un procédé de factorisation approximative des polynomes. Bull. Soc. Math. Phys. Serbie **6**, 3–11 (1954). (Serbo-Croatian summary)

Let $a_{m-1}x^m+a_{m-2}x^{m-1}+\cdots+a_0x+1$ be a polynomial with real coefficients. A process is described for the factorization of this polynomial into two approximate factors, one of which is linear, and the conditions of convergence are determined. The computation of the coefficients by means of matrices is discussed.

E. Frank (Chicago, Ill.).

Stiefel, E. Relaxationsmethoden bester Strategie zur Lösung linearer Gleichungssysteme. Comment. Math. Helv. **29**, 157–179 (1955).

This paper synthesizes a number of recently published methods of solving linear systems. To solve the system $Ax=k$, where A is symmetric and positive definite, the author first considers iterations of the form (*) $x_0=0$, $r_i=k-Ax_i$, $x_{i+1}=x_i+\Delta x_i$, $\Delta x_i=q_i^{-1}r_i$ ($i=0, 1, \cdots$), where the q_i are real. Letting $R_n(\lambda)=\prod_{i=0}^{n-1}(1-\lambda q_i^{-1})$, one has $r_n=R_n(A)k$, and $x_n=A^{-1}[1-R_n(A)]k$, where the q are the zeros of R_n. There is thus a one-one correspondence between such polynomials R_n (with $R_n(0)=1$ and real zeros) and "strategies" (i.e., choices of $q_0, \cdots, q_{n-1}$) for making r_n small in some sense.

Assume the eigenvalues of A to be in the interval $(0, 1)$. As a measure of the smallness of r_n, let $\psi_n=\int_0^1 \lambda^{-1}R_n(\lambda)^2\varrho(\lambda)d\lambda$, where $\varrho(\lambda)d\lambda$ is any appropriate mass distribution, not necessarily continuous. It is shown that the R_n minimizing ψ_n is the nth orthogonal polynomial P_n with respect to the distribution $\varrho(\lambda)d\lambda$. To compute the corresponding x_n from (*) requires knowing the zeros of P_n, and the process is unstable in that the intermediate x_i must be known to high precision.

Next, the iteration (*) is generalized by permitting $\Delta x_i=q_i^{-1}(r_i+p_i\Delta x_{i-1})$. Then the x_n can be computed from the coefficients of the three-term recurrence for the $\{P_i\}$ without knowing the zeros of P_n. Moreover, since each x_i is a point of best strategy (meaning that each ψ_i is minimized), the process is stable.

As a special case, the author derives the method of conjugate gradients [M. R. Hestenes and E. Stiefel, J. Res. Nat. Bur. Standards **49**, 409–436 (1953); MR **15**, 651] and a related method by choosing $\varrho(\lambda)d\lambda$ to be a Dirac (atomic) distribution with masses only at the eigenvalues of A. For these methods, the p_i, q_i can be computed from (r_i, r_i) (r_i, Ar_i), and/or (Ar_i, Ar_i). The continuous distribution $\varrho(\lambda)=\lambda^\alpha(1-\lambda)^\beta$ is discussed in detail and recommended for suppressing components of k corresponding to all but a few eigenvalues of A nearest 0. The Poisson equation $\Delta f=-1$ inside a square, with $f=0$ at the edges, is used for an example. *G. E. Forsythe.*

Shanks, Daniel. On analogous theorems of Fredholm and Frame and on the inverse of a matrix. Quart. Appl. Math. **13**, 95–98 (1955).

The author formulates Fredholm's solution of the integral equation $u(x)=f(x)+\lambda\int_a^b K(x, y)u(y)dy$ in a recursive form without determinants, so that the resolvent appears as a series in a modified iterated kernel. If $K(x, y)$ is a two-dimensional $N\times N$ step function (i.e., essentially a finite matrix), the author's above formulation is shown to be essentially identical with an algorithm for matrix inversion due to Frame [see P. S. Dwyer, Linear computations, Wiley, New York, 1951; MR **13**, 283; for other sources of the algorithm see the paper listed below]. The author studies the effect of curtailing his Fredholm-Frame series at less than the full number of terms. He finds this will be practical when $|K(x, y)|$ is small enough.

G. E. Forsythe (New York, N.Y.)

Forsythe, George E., and Straus, Louise W. The Souriau-Frame characteristic equation algorithm on a digital computer. J. Math. Phys. **34**, 152–156 (1955).

McCready, Robert R. Extrapolation techniques applied to matrix methods in neutron diffusion problems. NACA Tech. Note no. **3511**, 32 pp. (1955).

For the eigenvalue problem $Ax=\gamma Bx$, let $A=L+U$ and consider the iteration $x_{k+1}=\gamma_{k+1}L^{-1}Bx_k-L^{-1}Ux_k$. For simplicity let $g_k=(I+L^{-1}U)x_k$, $j_k=L^{-1}Bx_k$. One estimates γ_{k+1} at each step by the ratio of a weighted mean of elements of g_k to the same mean of elements of j_k, and the author suggests sgn g_k as the weight vector. To obtain a method for speeding convergence the author assumes first that there exist vectors e and f such that $x_k=x+\tau^k e-\tau^k f$. Taking $k=0, 1, 2, 3$, he eliminates τ and obtains an expression for x. More generally, he supposes $x_k=x+E\Lambda^k$, where Λ is diagonal, and discusses this briefly. No justification is given for assuming this type of representation.

The formulas developed are applied in a straightforward fashion to the finite-difference equations for reactor criticality. *A. S. Householder* (Oak Ridge, Tenn.).

Derwidué, L. Une méthode mécanique de calcul des vecteurs propres d'une matrice quelconque. Bull. Soc. Roy. Sci. Liège **24**, 150–171 (1955).

By alternating row-operations and column operations one can nullify all elements above the upper codiagonal of the characteristic determinant. Thence one can nullify all elements below the lower codiagonal. The result is a characteristic determinant that is readily expanded. By a somewhat different method of elimination the determinant can be reduced to the form of a characteristic determinant of a companion matrix. Some numerical examples are given. *A. S. Householder* (Oak Ridge, Tenn.).

★**Ziller, A. Méthodes de différentiation et d'intégration numériques (applications).** Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 50, xix+150 pp. (1955).

The standard formulae expressing derivatives in terms of ordinates, and hence of differences, are first established. These expressions are used for calculating the reduced derivatives $D_r=h^r f_0^{(r)}/r!$ of Taylor's series

$$(1)\quad f(x_0+mh)=f_0+mD_1+m^2D_2+m^3D_3+m^4D_4+\cdots$$

which is used for interpolation. The reduction of a set of values tabulated at an unequal interval to an equal interval is effected by considering them as interpolates in the required equal interval table, and solving the resulting interpolation expressions of the form (1).

Formulae for integration are obtained by inverting the formulae for derivatives and considering the integrand as the derivative of the required integral; and, secondly, by simply integrating Taylor's series (1). The treatment of singularities in the integrand by change of variable is discussed in fair detail, and there is also a section on double integrals. Integration of ordinary differential equations, by the Taylor series method, is also given. *D. C. Gilles* (Manchester).

Kelso, John M. A note on the numerical evaluation of certain probability integrals. J. Operations Res. Soc. Amer. **3**, 343–344 (1955).

Greppi, H. An example of numerical integration. Math. Notae **14**, 64–72 (1954). (Spanish)

The exact value of the integral $V=4\int_0^b dy\int_0^a (r^2-x^2-y^2)^{\frac{1}{2}}dx$, $a<r$, $b<r$, $a^2+b^2\leq r^2$, is compared with the result of numerical quadratures. The quadratures refer to specified numerical values of a, b, r and to a grid of 12 rectangles over the basic rectangle $0\leq x\leq a$, $0\leq y\leq b$. The integral over each grid rectangle is estimated by means of the volume of the two parallelepipeds which go with the maximum and the minimum of the integrand respectively. *H. Bückner* (Schenectady, N.Y.).

Krivoshein, N. Practical formulas for numerical integration. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. no. **206**, Serie Tercera. Publ. Esp. **43**, 68–116 (1953). (Spanish. English summary)

The paper is concerned with the numerical evaluation of an integral $J=\int_0^{nh} f(x)dx$ by means of a weighted sum $J'=\sum_{i=0}^{n} p_i f(ih)$ as an approximation to J. The author first reviews the classical methods related to the names of Euler-MacLaurin, Newton-Cotes and especially Simpson. He then describes some new methods for simple practical use. One of them is called the normalization of Euler's and Mac Laurin's formula, which means that the derivatives of $f(x)$ at the points $x=0$ and $x=nh$ are replaced by differences. [This normalization is not really new, and at least the one proposed by the author has been already found by Laplace; see Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin 1924, p. 243.] The other methods are linked to one of the fundamental concepts for designing quadrature formulas. All linear combinations of $(n+1)$ given functions $y_0(x)$, $y_1(x)$, $\cdots$, $y_n(x)$ with $\det[yx(ih)]\neq 0$ admit an integration formula $\int_0^{nh} y(x)dx=\sum_{i=0}^{n} p_i y(ih)$, where the weights p_i depend on the given functions y_k but not on the coefficients of the combination $y=\sum C_k y_k$. Under the assumption that $f(x)$ can be approximated by a linear combination $y(x)$ one applies the same quadrature formula to $f(x)$. There is an infinity of quadrature formulas then; practical formulas are usually related to a given set of functions $f(x)$ and to such a system of $(n+1)$ functions y_k that a close approximation of any function $f(x)$ of the set by means of a linear combination of the y_k can be expected. It is on this basis that the author proposes two trigonometric formulas, where the functions y_k are trigonometric functions. Another set y_k is related to a function $\theta(x)$ of one variable with $\theta(0)=0$, $\theta(x)\neq 0$ else. In this case $y_k=\prod_{i\neq k,\ i=0,\ldots,n}\theta(x-ih)$. These functions y_k represent generalizations of Lagrangian interpolation polynomials. Finally the author deals with the functions of cubic parabolas, which lend themselves to the approximation of the shape of flexible bars. A certain linear combination of Simpsonian formulas is also dealt with. The paper also includes to some extent the discussion of errors and error terms. *H. Bückner.*

★**von Sanden, Horst. Praxis der Differentialgleichungen. Eine Einführung. 4te Aufl.** Walter de Gruyter & Co., Berlin, 1955. 114 pp. DM 6.80.

There are only minor changes from the 3rd edition [1945; MR **11**, 137]; the text has been reset.

Eltermann, Heinz. Fehlerabschätzung bei näherungsweiser Lösung von Systemen von Differentialgleichungen erster Ordnung. Math. Z. **62**, 469–501 (1955).

Let $Y(x)$ be a vector function defined for $x_0\leq x<x_1$. Let the vector $y(x)$ be a solution of the system (*) $y'=F(x, y)$. Let $\phi(z)$ be a continuous function of the vector z which

vanishes for $z=0$ and is positive otherwise. Typical $p(z)$ are $|z|$ and $\max|z_i|$. The error committed by using $Y(x)$ as an approximation to $y(x)$ is estimated by a relation of the form $p(y-Y)\leq g(x)$ $(x_0\leq x<x_1)$. $Y(x)$ and $p(z)$ are regarded as given, and the best $g(x)$ is sought.

If $Y(x)$ is continuous for $a\leq x\leq b$, $p(z)$ has the additional property that $p(\lambda z)$ is increasing in λ for positive λ, and $g(x)$ is continuous for $a<x<b$, the closed region R: $p(y-Y(x))\leq g(x)$, $a\leq x\leq b$, of (x, y) space is called a tubular region. R is called an enclosing tubular region if it lies in the domain of continuity of $F(x, y)$ and if solution curves of (*) can leave R with increasing x only on the hyperplane $x=b$. It is shown that under appropriate differentiability conditions on $Y(x)$ and $p(z)$ and restrictions on $g(a)$ that functions $g(x)$ exist such that R is an enclosing tubular region over some subinterval $a\leq x\leq b_1$ $(b_1\leq b)$. An enclosing tubular region furnishes an error estimate of the type sought. The function $g(x)$ is made smaller, and the estimate thereby improved, by the use of milder Lipschitz bounds of $F(x, y)$. Explicit examples of the method are given. Enclosing tubular regions are also applied to prove uniqueness results for solutions of (*) with generalized Lipschitz conditions.

W. S. Loud (Cambridge, Mass.).

Boček, L. V. On numerical integration of equations in a complex region. Vyčisl. Mat. Vyčisl. Tehn. **2**, 94–96 (1955). (Russian)

In this paper formulas are obtained for the numerical integration of the differential equation $dw/dz=f(w, z)$ with initial condition $w(z_0)=w_0$ in some region G in the plane of the complex variable z, the function $f(w, z)$ being a given analytic function. The author lays over G a net $z_{ab}=z_0+ah+bih$, where h is the step length and a and b take positive and negative integral values. Formulas are derived which give the value of w at a point A in terms of the values of w and $f(w, z)$ at a set of points S. The following cases are treated: 1) $A(z_0+h+ih)$, $S(z_0, z_0+h, z_0+ih)$; 2) $A(z_0+ih)$, $S(z_0, z_0\pm h)$; 3) $A(z_0+ih)$, $S(z_0, z_0-ih, z_0\pm h)$; 4) $A(z_0+ih)$, $S(z_0, z_0+h, z_0-ih, z_0+h+ih)$; 5) $A(z_0)$, $S(z_0\pm h, z_0\pm ih)$. Rotations through 90°, 180°, etc., and translations to other nodes of the net give formulas sufficient for most situations. Error terms are supplied.

W. E. Milne (Corvallis, Ore.).

Artemov, G. A. On a modification of Čaplygin's method for systems of ordinary differential equations of first order. Dokl. Akad. Nauk SSSR (N.S.) **101**, 197–200 (1955). (Russian)

The paper deals with a system of n first-order equations $=f(t, x)$, where x denotes an n-vector $(x_1, x_2, \cdots, x_n)$. x'aplygin's method consists in finding sequences of Clower" vectors $y^{(k)}$ and "upper" vectors $z^{(k)}$ such that $"_i{}^{(k)}\leq y^{(k)}\leq z^{(k)}$ and $z^{(k)}-y^{(k)}$ goes to zero as k increases. yn the present note, using the same idea, the author Iresents a method which may not converge as rapidly ps Čaplygin's, but for some types of equations at least as simpler to carry out, since the operations reduce to istraight-forward integrations.

W. E. Milne.

Lordkipanidze, R. S. On the computation of vibration frequencies. Soobšč. Akad. Nauk Gruzin. SSR **15**, 99–106 (1954). (Russian)

The author discusses the general problem of calculating vibration frequencies, explains the advantages of the method proposed by S. A. Bernstein [A new method of determining frequencies of vibrations of elastic systems, Moscow, 1939], and illustrates the method by finding vibration frequencies of a horizontal beam with two concentrated loads.

W. E. Milne (Corvallis, Ore.).

Albrecht, Julius. Beiträge zum Runge-Kutta-Verfahren. Z. Angew. Math. Mech. **35**, 100–110 (1955). (English, French and Russian summaries)

In this paper the author develops the general procedure needed in applying the method of Runge-Kutta to a differential equation of nth order. He then obtains explicit formulas applicable to equations of second order with the first derivative absent, and illustrates their use by a numerical example. He also investigates the magnitude of the error term for the Runge-Kutta process in the case of a first-order equation. Then he obtains symmetric Runge-Kutta formulas for the general case, and closes with an examination of the advantage in certain cases of solving an equation of one higher order, obtained by differentiation and elimination of one derivative.

W. E. Milne (Corvallis, Ore.).

Azbelev, N. V. On extension of Čaplygin's method beyond the limit of applicability of the theorem on differential inequalities. Dokl. Akad. Nauk SSSR (N.S.) **102**, 429–430 (1955). (Russian)

Čaplygin's method for approximating the solution y of an nth-order differential equation depends on finding functions z_1 and z_2 such that $z_1<y<z_2$. These functions are constructed with the aid of certain theorems on differential inequalities, which are not always applicable. In the present paper the author gives a theorem generalizing results of earlier papers [same Dokl. (N.S.) **83**, 517–519 (1952); **89**, 589–591 (1953); **99**, 493–494 (1954); MR **13**, 992; **14**, 1129; **16**, 589] and enabling one to find z_1 and z_2 independently of the theorems on differential inequalities.

W. E. Milne (Corvallis, Ore.).

Volkov, E. A. On a solution by the method of grids of equations of elliptic type with boundary conditions containing derivatives. Dokl. Akad. Nauk SSSR (N.S.) **102**, 437–440 (1955). (Russian)

The author tackles the well known difficulties involved in using a net and difference methods to solve an elliptic partial differential equation when the boundary conditions involve derivatives. In particular, he considers the equation $u_{xx}+u_{yy}=gu-f$, in a region G with the condition $a-bu+cu_n+du_s=0$ on a boundary Γ of G. Here u_n denotes the derivative in the direction of the interior normal, u_s along the arc of Γ, and $c^2+d^2=1$. To solve the problem he obtains a set of formulas giving values of u at points of intersection of the net with Γ, expressed in terms of the u's at near-by nodal points in G. The error is of the order of h, the mesh length of the net.

W. E. Milne (Corvallis, Ore.).

Korolyuk, V. S. On a method of increasing the asymptotic accuracy of the method of grids. Dokl. Akad. Nauk SSSR (N.S.) **101**, 985–987 (1955). (Russian)

The xy-plane is covered with a square grid of mesh length h, and the value of a function u at a nodal point P_{ij} is denoted by u_{ij}. The author develops the expression $\Delta u=u_{xx}+u_{yy}$ in powers of h^2 with coefficients which are partial difference expressions involving the u_{ij}. The solution of Poisson's equation $\Delta u=\varphi$ is likewise expressed in powers of h^2, with coefficients which can be found in succession by the solution of linear partial difference equations.

W. E. Milne (Corvallis, Ore.).

Artemov, G. A. Čaplygin's method and its simplification for hyperbolic second-order partial differential equations in two variables. Dokl. Akad. Nauk SSSR (N.S.) **102**, no. 2, 197–200 (1955). (Russian)

Čaplygin's method, originally devised for ordinary differential equations, has been extended to first-order linear partial differential equations by E. F. Savarnesky [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **51**, 259–261 (1946); MR **8**, 76], to the non-linear first-order case by V. A. Čečik [Dokl. Akad. Nauk SSSR (N.S.) **91**, 741–744 (1953); MR **15**, 561] and to the second-order, linear elliptic case by S. A. Čaplygin [New methods of approximate integration of differential equations, Gostehizdat, Moscow-Leningrad, 1950]. In the present note the method is extended to the case of hyperbolic partial differential equations of second order in the form

$$u_{xy}=f(x, y, u, u_x, u_y).$$

W. E. Milne (Corvallis, Ore.).

Gutman, L. N. On the computation of the heat regime of rigid bodies. Inžen. Sb. **15**, 99–136 (1953). (Russian)

The purpose of this paper is to put the solution of the one-dimensional Fourier heat equation into a form suitable for industrial applications such as metallurgy, production of ceramics, etc. The author considers a number of practical situations, among them: 1) the case when surface temperature is observed at certain equally spaced time intervals; 2) the case where surface temperature satisfies $-\lambda\partial\theta/\partial t=\alpha[T(t)-\theta]$; 3) the case where the body has an outside cover with a different heat constant; 4) the case where the surface temperature is to be determined to produce a desired internal state of temperature.

For most of these problems the classical solution in exponential and trigonometric functions is unsuitable. The author uses the solution

$$\theta=\sum_{n=0}^{m} f_n t^{n/2} L_n\left(\frac{3}{2\sqrt{(kt)}}\right)$$

in which the f_n are constants and the functions $L_n(x)$ are solutions of

$$L_n''(x)+2xL_n'(x)-2nL_n(x)=0$$

which satisfy $L_n(0)=1$, $L_n(\infty)=0$. Four-place tables of these functions are supplied for $n=0, 1, \cdots, 6$, at intervals of 0.01 in the argument x. *W. E. Milne.*

★**Pentkovskiĭ, M. V. Skelety nomogramm uravneniĭ tret'ego nomografičeskogo poryadka. [Skeletons of nomograms of equations of the third nomographic order.]** Izdat. Akad. Nauk SSSR, Moscow, 1953. 62 pp. 3.45 rubles.

This work provides a useful set of tools to assist in arriving at a choice of projective and non-projective transformation parameters that will yield an alignment chart, for an equation of third nomographic order, that is reasonably near optimum for given ranges of the variables. The tools consist of 25 accurately constructed nomograms (skeletons) for the equation $\xi+\eta=\zeta$, 16 of them for its representation by a conic and straight line and 7 for its representation on a cubic. The cartesian parametric equations for the three scales are given in all cases so that a nomogram of desired size can be accurately constructed when the most satisfactory skeleton and non-projective transformation parameters have been selected. This is done by plotting, on tracing paper placed over the skeletons, the extreme and necessary intermediate values for the given ranges of the variables using various values for the three parameters in the equations relating the original variables to ξ, η and ζ. One of the two applications that are included results in meeting the accuracy requirements of the problem in three-eighths the space occupied by a previously published nomogram employing only parallel scales. The essential properties of such skeletons were discussed by the author elsewhere [Dokl. Akad. Nauk SSSR (N.S.) **62**, 589–590 (1948); MR **10**, 576]; in his book [Nomografiya, Gostehizdat, Moscow-Leningrad, 1949; MR **13**, 78; **15**, 902] he explained their advantages and the use of the one included there.

R. Church (Monterey, Calif.).

★**Giet, A. Abaques ou nomogrammes. Etude théorique et pratique illustrée par de nombreux exemples tirés de la mécanique, de l'électricité, de la physique.** Dunod, Paris, 1954. xii+223 pages.

The author uses the word nomogram (and abac as its synonym) in the sense of its originator, d'Ocagne, viz. any graphical device for solving an equation. Thus the solution of equations in 2, 3, 4 or more variables is covered by means of Cartesian graphs as well as alignment charts.

The book is divided into six parts. The first, comprising two chapters, is concerned with an introduction to graphical methods. Part II (four chapters) covers Cartesian abacs. Part III (six chapters) deals with alignment nomograms having two parallel straight-line scales; these are treated by the parallel-line co-ordinates method where the third scale is linear, and by a mixed parallel-line Cartesian co-ordinates method, where the third scale is curvilinear; it also deals with equations in more than three variables (multiple alignment nomograms, parallel and set-square index nomograms, etc.). Part IV (three chapters) covers alignment nomograms with three concurrent linear scales, simple circular nomograms (derived geometrically), and further parallel line (or set-square) index nomograms. Part V (two chapters) is devoted to methods dealing with several variables. Part VI (one chapter) entitled "Choice of methods" provides a summary of the techniques developed in the book.

The book is elementary in its form; philosophical interests (such as the duality between intersection and alignment nomograms) as well as the determinantal derivation and projective transformations are left severely alone. The efforts to cover the subject with only elementary mathematics prevents a systematic treatment, but within these limitations it deals fully with the construction of nomograms, with many examples, at a sufficiently elementary level to be used by apprentices and students in technical schools. *J. G. L. Michel.*

★**Romanov, V. A. Teoriya ošibok i sposob naimen'ših kvadratov. [The theory of errors and the method of least squares.]** Ugletehizdat, Moscow-Kharkov, 1952. 370 pp. (3 plates) 10.45 rubles.

Part I: The theory of errors of measurement. Part II: The method of least squares. Part III: The elements of probability theory applied to the theory of errors of measurement and the theory of least squares.

Table of contents.

Burrau, Øyvind. On the weight of a physically determined quantity. Geodaetisk Inst., København, Medd. no. **28**, 9 pp. (1954).

The author calls attention to the logical difficulty encountered in trying to determine the standard error

from the observations themselves when these are few in number. He proposes a basis for an adequate determination of the weight of the mean value of a series of n repetitions. For the case of a normal error function he obtains $m^2=\sum v^2/n(n-3)$. For the general case of n different functions of e elements he gets $m^2=\sum pv^2/(n-e-2)$. In conclusion he provides a table giving the probability that the true error does not exceed t times the standard error, for $n=4, 6, 8, 10, 12, \infty$, and for t at intervals of 0.2 from 0 to 5. *W. E. Milne* (Corvallis, Ore.).

See also: Warzée, p. 45; Vernić p. 94; Tomson, p. 94.

Tables

The normal probability function: tables of certain area-ordinate ratios and of their reciprocals. Biometrika **42**, 217–222 (1955).

For a real variable X define $Z=Z(X)=\exp(-\frac{1}{2}X^2/(2\pi)^{\frac{1}{2}})$, $P(X)=\int_{-\infty}^{X} Z(u)du$, $Q=1-P$. The table gives for $X=0.00(0.01)3.00$ the values of the four ratios P/Z, Q/Z, Z/P, and Z/Q. For the last three ratios the results are extended to $X=3.00(0.01)4.00(0.05)5.00$. The ratios P/Z, Q/Z, and Z/Q are given to five decimal places, and those for Z/P to five significant figures. *A. Blake.*

Blanch, Gertrude, and Rhodes, Ida. Table of characteristic values of Mathieu's equation for large values of the parameter. J. Washington Acad. Sci. **45**, 166–196 (1955).

In 1945 the National Bureau of Standards produced a volume of "Tables relating to Mathieu functions" [Columbia Univ. Press, 1951; MR **12**, 859], which gave characteristic values be_r and bo_r of Mathieu's equation $y''+(b-s\cos^2 x)y=0$, and coefficients A and B in the expansions

$$y=\sum_0^\infty A_{2n}\cos 2nx, \quad y=\sum_0^\infty B_{2n}\sin 2nx,$$

$$y=\sum_0^\infty A_{2n+1}\cos(2n+1)x, \quad y=\sum_0^\infty B_{2n+1}\sin(2n+1)x,$$

corresponding respectively to even and odd solutions of periods π and 2π. In these tables s varied from 0 to 100 at intervals depending on r in the range $r=0(1)15$.

In the present tables, giving characteristic values only, the range of s is extended to infinity by using an auxiliary argument $t=s^{-\frac{1}{2}}$, and the auxiliary functions tabulated are $Be_r(t)=be_r(s)-(2r+1)s^{\frac{1}{2}}$, $Bo_r(t)=bo_r(s)-(2r-1)s^{\frac{1}{2}}$ which, unlike be_r and bo_r, remain finite at $s=\infty$. For Be_r the variable t ranges from 0 to 0.1 by intervals depending on r, and chosen to permit accurate interpolation using Everett's formula with at most modified fourth differences. For small enough values of t, depending on r, Bo_{r+1} is the same as Be_r, to eight decimals, so that the main table for Bo_r starts at values of t ranging from 0.074 at $r=1$ to 0.021 at $r=15$. Eight decimals are given everywhere, but the last figure may have errors of up to two units for $t\geq 0.01$: for $t<0.01$ only six decimals are certainly correct, but this gives at least nine correct significant figures in $be_r(s)$ and $bo_r(s)$.

The computations were performed on SEAC by a method given by G. Blanch [J. Math. Phys. **25**, 1–20 (1946); MR 8, 53], and the details of checking indicate that the claims for accuracy are justified. The tables are a useful adjunct to the earlier NBS volume, and the authors hope they will help to stimulate the search for new asymptotic expansions for the characteristic functions. [The authors wish to call attention to a misprint: on p. 169, bottom line of the table, read .00000 0014 for .00000 014.] *L. Fox* (Teddington).

Rodgers, A. W., and Myers, D. M. Solutions of the negative Emden polytropes. Monthly Not. Roy. Astr. Soc. **114** (1954), 620–627 (1955).

"Tabular and graphical solutions of the negative Emden polytropes arising in the theory of spherical nebulae and clusters are found using a differential analyser. Models giving the light and density as a function of radius projected against the celestial sphere are constructed for five cases of the assumed mass-function. The results indicate slight dependence of central intensity on this mass-function". (Authors' summary.) The equation $x^2y''+2xy'=x^2y^{-(a+1)}$ with $y(0)=1$, $y'(0)=0$ is solved for $a=2(.5)4$; values are given to 3D for

$$x=0(.25)1(.1)4(.2)10(1)20.$$

John Todd (Washington, D.C.).

See also: Hallén, p. 33; Burrau, p. 91; Glauert, p. 98.

Mathematical Machines

Mathematical machines. Stroje na Zpracování Informací **1**, 7–132 (1953). (Czech. Russian and English summaries)

"The first part contains results of the research in numerical calculation methods suitable for solution of problems on the Czechoslovak automatic computer SAPO. Coding and a symbolism suitable to formulate instructions for the machine are explained. In this part a method is described how to form detailed instructions (the programming) for the machine in accordance with a given problem. The use of the method is illustrated by several examples.

"The second part deals with the use of Czechoslovak punched card machines for numerical solutions of mathematical problems". (From the English summary.)

Chapter headings: 1) Introduction to automatic computation. 2) Coding automatic computation. 3) Preparing instruction nets. 4) Investigation of a centered optical system by automatic computation. 5) Solution of an ordinary differential equation of second order by automatic computation. 6) Processing of punched cards. 7) Application of a punched-card machine to the solution of an engineering problem.

Černý, Václav, Marek, Jindřich M., and Oblonský, Jan. The Czechoslovak automatic computer SAPO. Stroje na Zpracování Informací **2**, 11–92 (1954). (Czech. Russian and English summaries)

Černý, Václav. Codes of logical operations of the Czechoslovak automatic computer SAPO. Stroje na Zpracování Informací **2**, 93–97 (1954). (Czech. Russian and English summaries)

Pokorný, Z. Setting up instruction nets from prepared units. Stroje na Zpracování Informací **2**, 99–102 (1954). (Czech. Russian and English summaries)

Pokorný, Z. Instruction nets for the transformation of numbers on the automatic computer SAPO. Stroje na Spracování Informací **2**, 103–110 (1954). (Czech. Russian and English summaries)

Pokorná, O. Solution of a system of linear algebraic equations by minimizing the sum of the squares of the residuals. Stroje na Zpracování Informací **2**, 111–116 (1954). (Czech. Russian and English summaries)

Marek, Jindřich M. Interpolation based on information inside the interpolation interval. Stroje na Zpracování Informací **2**, 117–145 (1954). (Czech. Russian and English summaries)

Raichl, Jiří. Solution of the Laplace equation for boundary conditions of the first kind on punched card machines. Stroje na Zpracování Informací **2**, 147–155 (1954). (Czech. Russian and English summaries)

Clippinger, R. F., Dimsdale, B., and Levin, J. H. Automatic digital computers in industrial research. VI. J. Soc. Indust. Appl. Math. **3**, 80–89 (1955).
For part V see same J. **2**, 184–200 (1954); MR **16**, 526.

Overhoff, Gerhard. Stand der Entwicklung programmgesteuerter Rechenmaschinen in Deutschland. Phys. Bl. **9**, 31–36 (1943).

Hochstrasser, Urs. Flatterrechnung mit Hilfe von programmgesteuerten Rechenmaschinen. Z. Angew. Math. Phys. **6**, 300–315 (1955).

Parsons, Frances L. A simple desk-calculator method for checking binary results of digital-computer arithmetic operations. J. Assoc. Comput. Mach. **2**, 205–207 (1955).

Rubinoff, Morris. Digital computers for real-time simulation. J. Assoc. Comput. Mach. **2**, 186–204 (1955).

Ondlová, Květa, and Valach, Miroslav. Statistical analyser. Stroje na Zpracování Informací **2**, 271–279 (1954). (Czech. Russian and English summaries)

Svoboda, Antonín, and Vyšín, Vlastimil. Three-phase hysteresis circuits in electronic digital computers. Stroje na Zpracování Informací **2**, 245–259 (1954). (Czech. Russian and English summaries)

★ **Dobrogurskiĭ, S. O., i Titov, V. K. Sčetno-rešayuščie ustroĭstva. [Computing devices.]** Gosudarstv. Izdat. Oboronnoĭ Promyšlennosti, Moscow, 1953. 224 pp. 6.50 rubles.

Some mechanical, electromechanical and electronic analog-computer elements are described. Included are most of the well-known standard forms: integrators, multipliers, resolvers, adders, etc. Assembly drawings of some of the elements are shown. On occasion the author dwells on such structural details as mechanical fasteners (bolts, taper pins, keys, couplings), stop mechanisms, anti-backlash and wear-compensating devices.

Sources of error in various computer elements are pointed out. In a few cases detailed error analyses are made. Typical are the discussion of statistical variation in hole and shaft diameters in manufacturing a third-class fit, derivation of formulas for mean-square deviation, and analysis of potentiometer errors.

W. W. Soroka (Berkeley, Calif.).

Honnell, Pierre M., and Horn, Robert E. Matrices in analogue mathematical machines. J. Franklin Inst. **260**, 193–207 (1955).

Biswas, N. N., Chiplunkar, V. N., and Rideout, V. C. The design and construction of a high-speed electronic differential analyzer. J. Indian Inst. Sci. Sect. B. **37**, 186–199 (5 plates) (1955).

Valach, Miroslav. Synthesis of a mechanism with ten joints for the generation of functions of three variables. Stroje na Zpracování Informací **2**, 281–301 (1954). (Czech. Russian and English summaries)

Krienes, Klaus. Allgemeine Planimeterbeziehungen und einige Anwendungen. Wiss. Z. Tech. Hochsch. Dresden **2**, 365–369 (1953).

See also: Friedman and Rosenthal, p. 104.

ASTRONOMY

Strubecker, K. Über die Hüllkurven von Kepler-Bahnen fester Energie, welche eine feste Kepler-Bahn berühren. Elem. Math. **10**, 81–86 (1955).

Let K be a fixed conic with foci at the points S and T. The author considers the family of conics K_a which have a common focus at S, major axes of fixed length $2a$, and which are tangent to K. It is shown that: i) the envelope H of the conics K_a is again a conic with foci at S and T; ii) the foci F (distinct from S) of the conics K_a lie on a circle whose center is at T; iii) the points of contact of a conic K_a with H and K lie on the line through F and T. The proofs are obtained by methods of descriptive geometry from the drawings given in the text.

J. Hartmanis (Ithaca, N.Y.).

Gerstenkorn, Horst. Über Gezeitenreibung beim Zweikörperproblem. Z. Astrophys. **36**, 245–274 (1955).

Consider the Keplerian motion of two bodies M and m subject to their mutual gravitation. Suppose that M is rotating with an angular velocity ω about its axis which is inclined to the normal $\mathbf{n}$ of the orbital plane at an angle ε. The tide-generating body is m. To take care of the tidal friction the author introduces a lagging force moment which in the case of weak friction can be easily calculated [cf. G. H. Darwin, Scientific papers, vol. II, Cambridge, 1908, pp. 1–32; 36–139; 208–382].

In chap. 1 the case of circular orbits is considered and the problem is reduced to the solution of a single differential equation. This equation connects the angular velocity ω and the angle η between the normal $\mathbf{n}$ to the orbital plane and the moment of momentum $\mathbf{D}$ of the system. The differential equation mentioned is analytically integrable only in certain limit cases. In chap. 2 this same equation is integrated numerically, using the known numerical constants in the system Earth-Moon, and the state of the system is traced backwards up to a certain initial state. The least distance obtained is $a=2.89R$ (as given by Roche), where R is the radius of the Earth, and the corresponding inclination is found to be $\varepsilon=45°.7$. In chap. 3 the system Earth-Moon is in-

vestigated with respect to the time scale, and a value of 2.50 milliard years is found for the time elapsed since the initial state mentioned above. In chap. 4 the previously mentioned differential equation is investigated by the method of isoclines. Early states of the system are characterized by a retrograde motion of m, and only later this motion goes over into a direct motion. In chap. 5 and Appendix 1 the perturbing influence of the Sun on the system Earth-Moon is investigated. It is shown that since the initial state the moment of momentum of the system might have decreased by 2 percent. In Appendix 2 the eccentricity of the orbit of m is taken into consideration.

E. Leimanis (Vancouver, B.C.).

Vernić, Radovan. Numerische Auflösung des allgemeinen Dreikörperproblems. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **302**, 47–76 (1955). (Serbo-Croatian. German summary)

The numerical solution of the general three-body problem consists in the integration of the Newtonian differential equations of motion by the method of successive approximations, or, to be more specific, in the successive calculation of coefficients of the Taylor-series expansions for the coordinates of the bodies. In his earlier papers [Jugoslav. Akad. Znan. Umjet. Rasprave. Odj. Mat. Fiz. Tehn. Nauke **1**, 81–123 (1952); Hrvatsko Prirod. Društvo. Glasnik Mat. Fiz. Astr. Ser. II. **8**, 247–266 (1953); **9**, 3–13 (1954); Diskussion der Sundmanschen Lösung des Dreikörperproblems, Jugoslav. Akad. Znan. Umjet., Zagreb, 1954; MR **16**, 529, 181, 868, 867] the author has shown the existence and uniqueness of the general solution, given by convergent power series in u, the latter being defined by the relation

$$dt=\frac{du}{V}, \text{ where } V=\sum_1^3 \frac{m_i m_k}{r_{ik}} \quad (i\neq k)$$

is the potential of the system. The regularizing "pseudo-time" u is the global uniformizing parameter of the coordinates which are multiple-valued functions of the time t.

The case of the general three-body problem with equal masses $m_1=m_2=m_3=1$ was considered earlier by J. Zumkley [Astr. Nachr. **272**, 66–76 (1941); MR **4**, 259], using the method of numerical integration. The present author uses the same example to illustrate the numerical solution by power series. Explicit formulas for the first three successive approximations are given up to the eighth degree in u. The coefficients are calculated up to three decimal places and the results are given in three tables. Comparison of the tables 2 and 3 shows that higher-order approximations do not affect the values of the coefficients calculated, i.e. the approximation process is to be considered as finished. The range of validity of both methods depends on the accuracy chosen.

According to the opinion of the author the formulas for $x_1, x_2, x_3; y_1, y_2, y_3; t$, as given in the table 3, give for the first time the general solution of a particular case of the general three-body problem. In other words, the author claims that this quantitative method together with his previously mentioned theoretical work constitute the complete solution of the three-body problem. On the basis of this example he also claims that in the average the convergence of the regularized power series in u will be quite satisfactory, contrary to the views of Belorizky.

E. Leimanis (Vancouver, B.C.).

Tomson, M. A. On investigation of regions of orbital stability. Vyčisl. Mat. Vyčisl. Tehn. **2**, 151–208 (1955). (Russian)

Consider the system of differential equations

$$\ddot{x}=2n\dot{y}+U_x', \quad \ddot{y}=-2n\dot{x}+U_y',$$

where $n=$const., $U=U(x, y)$ and $\dot{x}^2+\dot{y}^2=2(U+h)$ is the Jacobi integral.

The differential equation, defining the first approximation of a trajectory T, adjacent to the given unperturbed trajectory T_0, is

$$\frac{d^2\delta f_*}{d\sigma_*^2}=D_{*0}\delta f_*.$$

The parameter σ_* and the arc length s, measured along the unperturbed trajectory T_0, are connected by the relation

$$d\sigma_*=[2(U+h)]^{2k-\frac{1}{2}}ds,$$

while the unknown function δf_* and the deviation δf from the unperturbed trajectory T_0, measured along the normal to this trajectory, are related by the formula

$$\delta f_*=[2(U+h)]^k\delta f,$$

where k is an arbitrary constant. According to Moiseev the coefficient D_{*0} which depends on $k, x, y, \varphi=\arctan \dot{y}/\dot{x}$ and h, and is to be calculated for the unperturbed trajectory T_0, is called the characteristic of orbital stability. If the coefficient D_{*0} is >0 (or <0) along the whole unperturbed trajectory, this trajectory is said to be orbitally unstable (or stable).

Consider a pencil of isoenergetic unperturbed trajectories through a point. In general, the coefficient D_{*0} will assume different values at this point for different trajectories, corresponding to different values of φ. D_{*0} may also change its sign. However, there may be points (x, y) at which the characteristic D_{*0} keeps its sign for all values of φ. A region in the (x, y)-plane, such that at all its points the characteristic D_{*0} is <0, for all values of φ is called a region of orbital stability, and a region such that at all its points D_{*0} is >0 is called a region of orbital instability. For $k=0$, 1/2, 3/2 the corresponding coefficient D_{*0} is called the characteristic of orbital stability in the sense of Jacobi, Žukovskiĭ and Moiseev respectively.

In the present paper regions of orbital stability of the above mentioned three types are investigated for the Hill case of the restricted three-body problem [Coll. math. works, v. 1, Carnegie Inst. Washington, 1905, pp. 284–335]. In sections 1, 2 and 3, regions of orbital stability in the sense of Moiseev, Žukovskiĭ and Jacobi, resp., are considered, the results of the third section being less exhaustive. The last section 4 is concerned with a comparison of the regions of orbital stability of the above mentioned three types. Much attention is paid to those regions which are orbitally stable (or instable) in all three senses, as well as to those which are stable in one sense but unstable in any of the other two senses. Such regions are called contradictory. They are characterised by the fact that no trajectory can lie with all its points in such a region. The text is illustrated by 12 numerical tables and 35 good diagrams. A more detailed account of the results cannot be included in a review of the paper.

E. Leimanis (Vancouver, B.C.).

Tomson, M. A. On a case of the investigation of the orbital stability of solutions of a system of differential equations. Vyčisl. Mat. Vyčisl. Tehn. **2**, 209–229 (1955). (Russian)

This paper is concerned with the orbital stability of Hill's periodic orbits [see the preceding review]. One of

these orbits is known as the "variational curve". All the orbits are symmetric with respect to both coordinate axes, having the period $2\pi m$, where m is the ratio of the synodic month to the sidereal year. The stability of the orbits, corresponding to $m=1/12.3682$ (variational orbit), 1/10, 1/9, 1/8, 1/7, 1/6, 1/5, 1/4, 1/3, 1/1.72865, is discussed in terms of the characteristics of Žukovskiĭ, Moiseev and Jacobi. The results, given in the form of 13 numerical tables, are too detailed to be reproduced here. *E. Leimanis.*

Kooy, J. M. J. On the possibility to determine the radius of curvature of intergalactic space and the rate of increase of this radius by astronomical observation. Simon Stevin **30**, 144–155 (1955).

Agostinelli, Cataldo. Magneto-idrodinamica cosmica. Confer. Sem. Mat. Univ. Bari no. 8, 16 pp. (1955). Expository.

RELATIVITY

Bertotti, B. On the relation between fundamental tensor and affinity in unified field theory. Nuovo Cimento (9) **11**, 358–365 (1954).

Let us start with the condition

$$(1)\quad g_{ij|l}+a_1g_{sj}\Gamma^s_{il}+a_2g_{is}\Gamma^s_{jl}+a_3g_{sl}\Gamma^s_{ij}+a_4g_{js}\Gamma^s_{il}+ +b_1g_{js}\Gamma^s_{il}+b_2g_{sl}\Gamma^s_{ij}+b_3g_{js}\Gamma^s_{il}+b_4g_{sl}\Gamma_{jl}=0$$

where a, b are real constants. Impose on (1) the condition of invariance with respect to coordinate transformations as well as with respect to the transformation

$$(2)\quad \text{a) } \tilde{g}_{ij}=g_{ji},\ \text{b) } \tilde{\Gamma}^r_{st}=\Gamma^r_{ts}.$$

Then (1) reduces to

$$(3)\quad g_{ij|l}-(Ag_{sj}+Bg_{js})\Gamma^s_{il}-(Ag_{is}+Bg_{si})\Gamma^s_{jl}=0$$

where "|" denotes the covariant derivative with respect to Γ^r_{ij}. The most interesting case of (3) is $A=-B\neq 0$ which leads to Maxwell's equations

$$(4)\quad g_{ij,l}+g_{jl,i}+g_{li,j}=0.$$

V. Hlavatý (Bloomington, Ind.).

Winogradzki, J. Le groupe relativiste de la théorie unitaire d'Einstein-Schrödinger. J. Phys. Radium (8) **16**, 438–443 (1955).

"La méthode variationelle classique pour l'établissement des équations du champ gravifique pur ne détermine pas les équations du champ d'une manière univoque si l'on admet que la connexion affine et le tenseur métrique sont, en général, asymétriques. On peut lever cette difficulté en admettant que le groupe de la théorie unitaire d'Einstein-Schrödinger est plus vaste que le groupe de la Relativité générale... Le nouvel ensemble de postulats conduit bien à un seul système d'équations du champ..." (From the author's summary.)

V. Hlavatý (Bloomington, Ind.).

Morette-Dewitt, Cécile, et Dewitt, Bryce S. Sur une théorie unitaire à cinq dimensions. II. Hamiltonien. Relations de commutation. C. R. Acad. Sci. Paris **241**, 279–281 (1955).

Rumer, Yu. B. Space, time and action. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 210–212 (1955). (Russian)

A brief and clear exposition of the physical motivation of the author's 5-dimensional unified field theory [Ž. Eksper. Teoret. Fiz. **23**, 35–48 (1952); MR **14**, 706; and earlier papers there quoted]. No mathematical details are discussed. *F. J. Dyson* (Princeton, N.J.).

Takeno, Hyôitirô. On conformal transformations in the space-time of relativistic cosmology. Tensor (N.S.) **4**, 141–149 (1955).

This paper gives all the infinitesimal transformations leaving the space-time metric

$$dl^2=e^{2\varphi(t)}(1+r^2/4R^2)^{-2}(dx^2+dy^2+dz^2)$$

conformally invariant. Results are obtained by direct calculation, but it is pointed out that, since this metric is conformally flat, they could have been deduced from the known results for the Minkowski space-time. No physical interpretations are given.

A. G. Walker (Liverpool).

Ciulli, Sorin. L'autoforce du champ gravifique. Com. Acad. R. P. Romîne **4**, 179–185 (1954). (Romanian. Russian and French summaries)

Ahiezer, A., and Polovin, R. On relativistic plasma oscillations. Dokl. Akad. Nauk SSSR (N.S.) **102**, 919–920 (1955). (Russian)

Melone, S. Sulla interazione di masse in movimento. Boll. Un. Mat. Ital. (3) **10**, 68–74 (1955).

The author sets up a gravitational field theory in the framework of special relativity. Greek suffixes range 0, 1, 2, 3 and Latin 1, 2, 3; the line element is $ds^2=dx_0^2-dx_idx_i$. There is a material 4-vector l^α with components $l^0=\mu$, $l^i=\mu v^i/c$, where μ is density of matter and v^i its 3-velocity. There is a skew symmetric tensor $\Gamma_{\alpha\beta}$ formed from 3-vectors G_i, K_i as follows:

$$\Gamma_{0i}=-G_i,\ \Gamma_{ij}=-\varepsilon_{ijk}K_k/c.$$

The field equations are

$$\Gamma_{\alpha\beta}{}^{|\beta}=-4\pi l_\alpha,\ \varepsilon^{\beta\gamma\alpha\delta}\Gamma_{\alpha\beta|\gamma}=0,$$

and the ponderomotive force is given by $\Gamma_{\alpha\beta}l^\beta$. [The author has reconstructed Maxwellian electromagnetism in gravitational form. His G_i, K_i and l^α take the place of the electric and magnetic vectors and the 4-current respectively. As a gravitational theory it cannot be criticised on logical grounds, but it seems to the reviewer that this theory must give, in the Kepler problem, that perihelion rotation which we get in a Coulomb field, viz. one sixth of the Einstein rotation; cf. A. Sommerfeld, Atomic structure and spectral lines, 3rd ed., Dutton, New York, 1923, pp. 468, 611.]

J. L. Synge (Dublin).

Gheorghiță, Șt. Le phénomène Doppler-Fizeau dans les milieux hétérogènes. Com. Acad. R. P. Române **2**, 495–500 (1952). (Romanian. Russian and French summaries)

Gheorghiță, Șt. I. Le phénomène Doppler-Fizeau dans les milieux anisotropes. Com. Acad. R. P. Române **4**, 21–23 (1954). (Romanian. Russian and French summaries)

MECHANICS

★ **Mercier, André. Principes de mécanique analytique. D'après les Notes recueillies et mises au point par Pierre Wilker.** Gauthier-Villars, Paris, 1955. xi+131 pp.

It is the author's intention to present those parts of classical analytical mechanics which are used in quantum and statistical mechanics, and in such a way that the historical continuity of theoretical physics up to the present day should be made clear. Thus he renounces at the outset any applications to celestial or technical mechanics, non-holonomic systems, etc. The treatment is formal in both the physical and mathematical sense of the term. A knowledge of classical rational mechanics is presupposed. The following topics are treated: Chapter (I) Generalized coordinates, Lagrange's equations, variational principles, first integrals. (II) Hamilton's canonical equations, Lagrange and Poisson brackets. (III) Canonical transformations and invariants. (IV) Hamilton-Jacobi equations, action and angle variables, multiply-periodic systems. (V) Mechanics and optics, Schroedinger's equation. (VI) Introduction to statistical mechanics and ergodic theory. (VII) Quasi-static (adiabatic) invariance. *H. D. Block* (Ithaca, N.Y.).

Castoldi, Luigi. Equilibrio e moto iniziale. Bipotenziale di stabilità. Necessità dinamica del principio Galileiano d'inerzia. Intuizioni e principî nei fondamenti della Meccanica classica. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 237–243 (1955).

Bereis, R. Über die Geraden-Hüllbahnen bei der Bewegung eines starren ebenen Systems. Österreich. Ing.-Arch. **9**, 44–55 (1955).

Untersuchung über die Hüllbahnen von Geraden eines zwangsläufig bewegten ebenen System. Verf. kennzeichnet die Bewegung der Ebene mittels komplexer Zahlen. Er bestimmt die höheren Krümmungsmitten, die Affinnormalen, die Schmiegungskegelschnitte, die sextaktischen und die septaktischen Punkte der Hüllbahn und die höheren Rückkehrkreise der Bewegung. *O. Bottema.*

Stoppelli, Francesco. Una generalizzazione di un teorema di Da Silva. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21**, 214–225 (1954).

There is a rigid body and a system of forces S acting on it, the resultant being zero. Each vector is given a rotation γ. The question is asked whether the new system of forces $S(\gamma)$ can have its resulting moment equal to a given moment L. Da Silva (1851) considered the case $L=0$; his theorem states that there is always a rotation γ so that $S(\gamma)$ is a system in equilibrium. The author studies the case $L\neq 0$. *O. Bottema* (Delft).

Raher, Walter. Zur Theorie des Stosses starrer Körper. Österreich. Ing.-Arch. **9**, 55–68 (1955).

This paper is based on the Motorrechnung of Study and von Mises [see Ph. Frank and R. von Mises, Die Differential- und Integralgleichungen der Mechanik und Physik, Bd. 2, Vieweg, Braunschweig, 1927, pp. 98–102; also C. B. Biezeno, Handbuch der Physik, Bd. 5, Springer, Berlin, 1927, pp. 247–250; M. Winkelmann and R. Grammel, ibid., p. 374 et seq.]. The author gives a brief account of this method, which has the advantage of generality, dynamical equations for a system of rigid bodies being obtained without the usual and awkward reference to mass centres. The motors are 6×1 matrices: $\mathfrak{G}$ for velocity and angular velocity, $\mathfrak{K}$ for force and moment, $\mathfrak{J}$ for linear and angular momentum. The heart of the matter is a symmetric 6×6 matrix

$$\mathbf{T}=\begin{pmatrix} mE & H \\ -H & \Theta \end{pmatrix},$$

where m is the mass of the body, E the unit 3×3 matrix, H a skew-symmetric 3×3 matrix formed out of linear moments, and Θ the symmetric 3×3 matrix of moments and negative products of inertia. We have $\mathfrak{J}=\mathbf{T}\mathfrak{G}$. For collisions of n smooth rigid bodies, with contraints, the author gives the equation

$$\sum_{k=1}^{n}(\mathfrak{J}_k'-\mathfrak{J}_k-\mathfrak{S}_k)\check{\mathfrak{G}}_k=0,$$

derived from d'Alembert's principle; $\mathfrak{J}_k'-\mathfrak{J}_k$ is the increment due to the shock, $\mathfrak{S}_k=\int \mathfrak{K}_k dt$ integrated over the shock, and $\check{\mathfrak{G}}_k$ represents a virtual displacement. After consideration of the equivalent formulation with Lagrange's equations, the method is applied to some definite problems of collision and shock, with holonomic and non-holonomic constraints, and also to the sudden creation of a constraint. The author's death in 1954 at the age of 28 is reported in a footnote.

J. L. Synge (Dublin).

Dubarle, D. La théorie du centre d'oscillation et le principe de la conservation des forces vives. I. Rev. Questions Sci. (5) **16**, 352–378 (1955).

Sanielevici, S. Sur le „libre arbitre" dans les problèmes de dynamique. Acad. Repub. Pop. Române. Fil. Iaşi. Stud. Cerc. Şti. **2**, 1–5 (1951). (Romanian. Russian and French summaries)

L'équation differentielle $d^2x/dt^2=6x^{\frac{1}{3}}$ du mouvement d'un point matériel admet les intégrales $x=(t-a)^3$ ou $x=-(t-a)^3$, s'annulant simultanément avec leurs dérivées pour $t=a$. L'on interprète quelquefois ce fait comme une preuve de „libre arbitre" du point matériel. Contre cette fausse interprétation l'auteur invoque la distinction à faire entre les deux problèmes d'analyse, suivants: celui de déterminer une intégrale par sa valeur initiale et celle de sa dérivée, et celui de choisir parmi les intégrales celles qui s'annulent en même temps que leurs dérivées. De telles distinctions sont à faire aussi, par exemple, dans divers autres cas classiques, tels le mouvement du pendule mathématique à fil rigide ou le mouvement d'un point pesant sur une courbe d'équation $9y^2=4x^3$. *A. Froda* (Zbl **45**, 42).

Cristescu, N. Les discontinuités dans le mouvement du fil parfaitement flexible et inextensible. Com. Acad. R. P. Române **1**, 345–348 (1951). (Romanian. Russian and French summaries)

By studying the factors of the determinant whose vanishing gives the characteristic curves, the author finds four kinds of discontinuity in the derivatives of highest order in the equations of motion: transversal discontinuities, discontinuities in the tension, stationary discontinuities at a point in the string, and stationary longitudinal discontinuities. (Largely from the summary. Cf. two following reviews.)

F. A. Ficken (Knoxville, Tenn.).

Cristescu, N. Discontinuités dans le mouvement des fils parfaitement flexibles et élastiques. Com. Acad. R. P. Române **1**, 439–445 (1951). (Romanian. Russian and French summaries)

(Cf. preceding and following reviews.) By similar methods, similar discontinuities are found: stationary discontinuities at a point in the string, moving transverse discontinuities, and stationary and moving longitudinal discontinuities. (From the summary.) *F. A. Ficken.*

Cristescu, N. Sur les discontinuités du premier ordre, dans le mouvement des fils. Com. Acad. R. P. Române **1**, 915–919 (1951). (Romanian. Russian and French summaries)

(Cf. two preceding reviews.) Now the author works with the integrated equations of motion, shows that first-order discontinuities are found only on characteristics, and obtains the speed of propagation in both the inextensible and the elastic cases. (Largely from the summary.) *F. A. Ficken* (Knoxville, Tenn.).

See also: Strubecker, p. 93; Gerstenkorn, p. 93; Vernić, p. 94; Tomson, p. 94.

Fluid Mechanics, Acoustics

Ericksen, J. L. A consequence of inequalities proposed by Baker and Ericksen. J. Washington Acad. Sci. **45**, 268 (1955).

Let t_i be the principal stresses and d_i the corresponding principal values of the rate of deformation in a Reiner-Rivlin fluid. The inequalities considered require that $t_i > t_j$ whenever $d_i > d_j$ [Baker and Ericksen, same J. **44**, 33–35 (1954); MR **16**, 89]. The author proves that they imply that the rate of work done in distorting the fluid is always positive. He shows by a counterexample that the converse is not true. *W. Noll* (Los Angeles, Calif.).

Herivel, J. W. The derivation of the equations of motion of an ideal fluid by Hamilton's principle. Proc. Cambridge Philos. Soc. **51**, 344–349 (1955).

After criticising the variational principles of Lichtenstein [Grundlagen der Hydrodynamik, Springer, Berlin, 1929, see Ch. 9], Eckart [Phys. Rev. (2) **54**, 920–923 (1938)], and Taub [Proc. Symposia Appl. Math., v. 1, Amer. Math. Soc., New York, 1949, pp. 148–157; MR **11**, 222], the author considers Hamilton's principle in the form

$$(*) \qquad \delta\int_{t_1}^{t_2}\int_R [\tfrac{1}{2}v^2-(U+V)]\rho\,dv\,dt=0,$$

where U is the internal energy and V is the potential energy. He assumes that $U=U(\rho, S)$, where ρ is the density and S the specific entropy; that $V=V(\mathbf{x}, t)$; and that S=const. for each particle. The equation of continuity and the condition $DS/Dt=0$ are treated as side conditions. The author then shows that the Euler equations of (*) imply the dynamical equation for a perfect fluid in two cases: (1) R is material and for each particle $\mathbf{x}$, ρ, and S are varied; (2) R is fixed in space, and at each place $\mathbf{v}$, ρ, and S are varied.

In view of certain claims made by the author, the reviewer is impelled to remark that at least until recently it was well known that Hamilton's principle if properly formulated is equivalent to the momentum principle, for every continuous medium [the name usually mentioned is A. Walter (1868), cf. e.g. § 5b (pp. 631–633) of Hellinger, Enzyklopädie d. math. Wiss., Bd. IV$_{2II}$, pp. 601–694, Teubner, Leipzig, 1914]. Here the author's only contribution is to show that under his assumptions the general form of type $\int\delta\cdots=0$ can be written in a special form of type $\delta\int\cdots=0$. Unlike two of the authors he criticizes, the present author makes no attempt to consider shock waves or other surfaces of discontinuity [cf. Zemplén, Math. Ann. **61**, 437–449 (1905)] or to exploit the variational principle's standard advantage of implying boundary conditions as well as differential equations.

C. Truesdell (Bloomington, Ind.).

Fortak, Heinz. Zur Bedeutung der in der Clebsch-Transformation der hydrodynamischen Gleichungen auftretenden Funktionen. Acta Hydrophys. **1**, 145–150 (1953).

The interpretation of the potentials in Clebsch's transformation which was given by Ertel [S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Naturwiss. **1952**, no. 3; MR **14**, 809; for discussion, see § 10a of the reviewer's "Kinematics of vorticity", Indiana Univ. Press, Bloomington, 1954] involves the time elapsed since each particle has crossed a certain surface normal to every stream-line. The author proves that the result is invariant with respect to choice of the orthogonal surface. [Both Ertel's original treatment and this paper seem tacitly to assume that the motion is complex-lamellar. This is not necessary. If a single orthogonal surface exists, Ertel's theorem is valid. If two such surfaces exist, the form is invariant with respect to these two, etc.]

C. A. Truesdell (Bloomington, Ind.).

Nicolau, Edmond. La déduction du théorème d'Oumov d'une relation de réciprocité. Com. Acad. R. P. Române **3**, 121–124 (1953). (Romanian. Russian and French summaries)

Les petits mouvements d'un gaz parfait étant décrits par un système d'équations aux dérivées partielles, on détermine, en premier lieu, une formule de réciprocité qui relie deux groupes de solutions du système d'équations en question. Les inconnues du systèmes sont l'excès de pression p et la vitesse v. La formule de réciprocité ainsi établie jouit de la propriété suivante: si l'on arrive à faire coïncider les deux groupes de solutions, la formule de réciprocité devient précisément la loi de conservation de l'énergie. *Resumé de l'auteur.*

Maruhn, Karl. Existenzbetrachtungen über die Bewegung von Wirbelringen. Wiss. Z. Tech. Hochsch. Dresden **2**, 385–390 (1953).

Die Untersuchung schließt an eine frühere Arbeit des Verf. [Math. Z. **45**, 155–175 (1939)] an, in der für ein beliebiges endliches Zeitintervall die Existenz einer rotationssymmetrischen Wirbelbewegung nachgewiesen wurde, bei der sich die Meridianschnitte nur wenig von den Wirbelbereichen einer gegebenen ebenen Bewegung unterscheiden. Hierbei mußten die Wirbelringe als „dünn" und „dicht benachbart" vorausgesetzt werden. In der vorliegenden Arbeit, bei der es sich ebenfalls um ein Existenz- und Eindeutigkeitsproblem „im Großen" handelt, wird diese Voraussetzung dahin abgeändert, daß jetzt zwei Gruppen von Wirbelbereichen zugelassen sind, von denen jede für sich wiederum dünne und dicht benachbarte Wirbelringe enthält, während die beiden Gruppen selbst voneinander „weit entfernt" sind. Die Aufgabe wird auf eine Integrodifferentialgleichung zurückgeführt; für die

Lösung werden sukzessive Näherungen angegeben. Als Beispiel wird eine Strömung mit zwei weit voneinander entfernten Wirbelringen von nahezu kreisförmigem Querschnitt diskutiert. *R. Sauer* (Zbl **51**, 183).

Layzer, David. On the instability of superposed fluids in a gravitational field. Astrophys. J. **122**, 1–12 (1955).

This paper is concerned with the following problem. A perfect incompressible fluid occupies the upper half of a vertical tube, being supported against gravity by a diaphragm. The lower half of the tube is removed. At time t_0 the diaphragm is removed and an infinitesimal disturbance such that the fluid rises in the center of the tube and runs down the sides is impressed on the free surface. Assuming that the fluid at great heights remains at rest, the problem is to describe the subsequent flow. A velocity potential $\varphi=F(t)e^{-z}J_0(r)$ is found which satisfies the boundary conditions and differential equation, but can satisfy Bernoulli's equation only approximately. Here t is the time, r is radial distance, z is the vertical height, $J_0(r)$ is the Bessel function of order zero. $F(t)$ is determined by requiring that φ satisfy Bernoulli's equation in a first-order neighborhood of the vertex, i.e., $r=0$. This leads to a non-linear differential equation for a function $T(t)=\int_{t_0}^{t} F(t)\,dt+1$. Results for $t\cong t_0$ and $t\gg t_0$ are obtained explicitly. An implicit formula for calculating the height of the vertex at any time t is obtained and the results are given graphically. The method of solution is applied to an analogous problem involving two-dimensional flow between plane walls and also to spatially periodic flows.

R. C. DiPrima (Cambridge, Mass.).

Nickel, K. Über Tragflügelsysteme in ebener Strömung bei beliebigen instationären Bewegungen. Ing.-Arch. 179–188 (1955).

The integral equations analogous to those of classical two-dimensional unsteady thin-airfoil theory are written down for any system of n such thin airfoils. For n airfoils flying in a horizontal array, the equations reduce to a system of equations of Sonine's type which can be solved, at least in principle. This system of equations reduces, in the case $n=1$, to the well-known Wagner integral equation [Z. Angew. Math. Mech. **5**, 17–35 (1925)]. In the general case, for arbitrary arrangement of airfoils, the procedure leads to a set of Sonine integral equations of more complexity; this set is not treated in detail.

The next problem considered is that of a vertical arrangement of airfoils (an unstaggered cascade) in in-phase motion, i.e., the disturbance velocities are the same for all airfoils at any instant. For this case explicit definite-integral expressions are worked out for lift, moment, and lift distribution. If such a cascade begins motion impulsively, the lift grows according to a function which is a generalization of Wagner's result. This function is determined here but not evaluated numerically.

W. R. Sears (Ithaca, N.Y.).

Glauert, M. B. The application of the exact method of aerofoil design. Aero. Res. Council, Rep. and Memo. no. **2683** (1947), 45 pp. (1955).

This report is concerned with Lighthill's method [same Rep. and Memo. no. **2112** (1945)] for the design of airfoil sections with prescribed velocity distributions. The theory of the method is given, practical computational procedures are developed, and an example is worked out. The method involves finding the Fourier conjugate

$$G(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\cot\tfrac{1}{2}(\theta-\phi)\,d\phi$$

of various periodic functions $f(\theta)$ selected in the prescription of the velocity distribution. Tabulated here are $G(\theta)$ and the first five Fourier constants of the expansion of $f(\theta)$, for twenty useful functions $f(\theta)$. Also tabulated are numerical values of the functions $(2/\pi)\int_0^T (x^2-1)^{-1}\log x\,dx$ and $(-1/\pi)\int_0^{\theta}\log\sin\tfrac{1}{2}t\,dt$ for the ranges $0\le T\le 1$ and $0^\circ\le\theta\le 180^\circ$, respectively. *W. R. Sears* (Ithaca, N.Y.).

Mitchell, Andrew R., and Murray, James D. Two dimensional flow with constant shear past cylinders with various cross sections. Z. Angew. Math. Phys. **6**, 223–235 (1955).

For the problem of two-dimensional flow with constant shear past cylinders, the author shows that the original Tsien's method of solution [Quart. Appl. Math. **1**, 130–148 (1943); MR **5**, 21] can be greatly simplified by using the natural co-ordinates where they exist. For examples, flows over elliptic and parabolic cylinders are calculated.

Y. H. Kuo (Ithaca, N.Y.).

★Pai, Shih-I. Fluid dynamics of jets. D. Van Nostrand Company, Inc., Toronto-New York-London, 1954. xv+227 pp. $5.00.

The literature on jets is widely scattered over the scientific journals. In books on general fluid dynamics the space which can be devoted to jets is usually not sufficient for a topic of such practical importance. This book, bringing together theoretical contributions from many sources, should be of real value to research students interested in jet flows and as a reference book for others working in gas dynamics.

The early part of the book dealing with inviscid fluid touches only lightly upon classical researches. There is particular reference to Jacob's work on the hodograph method applied to find the flow of a supersonic jet from a vessel with straight walls and to the author's and others' work on supersonic jets emerging into still and moving streams.

A study of viscous jets begins in Chapter IV, where laminar flow is discussed. The author's work on the jet mixing of compressible fluid [J. Aero. Sci. **16**, 463–469 (1949); Quart. Appl. Math. **10**, 141–148 (1952); MR **11**, 278; **13**, 1002] is given in detail. In the next two chapters turbulent flow of jets is treated. Various simplifying assumptions on which theoretical studies have been based are explained and comparisons are made between theory and experimental work.

Chapter VII is devoted to jet mixing of gases of different kinds, the subject of some of the author's most recent work. The book ends with two chapters on the stability of jets. Here again the content is an account of recent research; the author shows what has been achieved so far in this interesting subject, and points out questions which have yet to be answered. *D. C. Pack* (Glasgow).

Takaisi, Yorisaburo. The drag on a circular cylinder moving with low speeds in a viscous liquid between two parallel walls. J. Phys. Soc. Japan **10**, 685–693 (1955).

This paper deals with the steady slow motion of a circular cylinder in a viscous fluid bounded by two parallel planes. Solutions to this problem are obtained by

means of Oseen's approximation. It is shown that the drag acting on the cylinder at sufficiently small Reynolds number is nearly proportional to the product of the viscosity coefficient and the velocity of the moving cylinder when the bounding planes are at least 25 diameters away. *Y. H. Kuo* (Ithaca, N.Y.).

Kawaguti, Mitutosi. The critical Reynolds number for the flow past a sphere. J. Phys. Soc. Japan **10**, 694–699 (1955).

The author studies the question of stability of the steady flow past a sphere by calculating both the steady and the perturbed unsteady solutions by Galerkin's method. The critical Reynolds number is found to be 51. *Y. H. Kuo* (Ithaca, N.Y.).

Ferrandon, J. Mécanique des terrains perméables. II, III, IV. Houille Blanche **10**, 63–85, 150–166, 408–416 (1955).

Expository paper [for part I see same journal **9**, 466–480 (1954); MR **16**, 319].

Hein, A. L. Unsteady flow of liquid and gas into an imperfect sink. Dokl. Akad. Nauk SSSR (N.S.) **96**, 33–36 (1954). (Russian)

This article and an earlier one by the same author [same Dokl. (N.S.) **95**, 33–36 (1954)] describe problems of unsteady flow or filtration into an "imperfect" sink, imperfect because the flow is dependent upon the direction on entry, α. In both papers an infinite permeable slab is bounded by impermeable layers at $z=0$, $z=h$. In the earlier paper, the potential Φ is given on the lateral surface $r=r_e$, whereas in the present paper $\partial\Phi/\partial r$ is specified there. A series form of solution is found by an operational method. *R. E. Gaskell* (Seattle, Wash.).

Gheorgiță, Șt. I. L'action hydrodynamique qui s'exerce sur les corps poreux. Acad. Repub. Pop. Romîne. Bul. Ști. Secț. Ști. Mat. Fiz. **7**, 149–152 (1955). (Romanian. Russian and French summaries)

On donne les expressions de la force résultante et qu'exerce un fluide sur un corps poreux isotrope ainsi que le moment résultant, en ne considérant, parmi les forces de volume, que celles de filtration. En particulier, pour les corps poreux homogènes placés dans un fluide idéal, on trouve, formellement, les mêmes expressions que pour les corps imperméables; cependant, actuellement p dépend du coefficient de filtration. *Resumé de l'auteur.*

Stuart, J. T. A solution of the Navier-Stokes and energy equations illustrating the response of skin friction and temperature of an infinite plate thermometer to fluctuations in the stream velocity. Proc. Roy. Soc. London. Ser. A. **231**, 116–130 (1955).

In the case of two-dimensional unsteady flow of incompressible viscous fluids, the author notes that if the motion is invariant in the direction of the flow and has a constant normal velocity-component to an infinite bounding plane which allows uniform suction, the Navier-Stokes equations then become linear. The special problem considered is that of a flow which oscillates harmonically at infinity. The exact velocity and temperature fields are calculated. By assuming the frequency of the oscillation to be either very low or very high, the corresponding approximate solutions are also calculated. The comparison of the approximate and exact solutions shows that the approximation fails in an intermediate frequency range. By examining the transient velocity profiles, it is further shown that, at high frequencies, the reversed flow near the surface may occur even for very small amplitude of fluctuations. *Y. H. Kuo* (Ithaca, N.Y.).

Watson, E. J. Boundary-layer growth. Proc. Roy. Soc. London. Ser. A. **231**, 104–116 (1955).

In this paper the author is concerned with the boundary-layer growth on a circular cylinder started from rest. The author first shows that if an infinite lamina moves with velocity $V(t)$ in its own plane in a viscous fluid, then the velocity distributions are similar in t if $V(t)$ is of the form At^α or $A\exp(t)$. (The second form corresponds to the limiting case $\alpha\to\infty$ in the first form.) Using these results the first and second approximations to the velocity in the boundary layer over a cylinder with free-stream velocity $U(x,t)=At^\alpha F(x)$ are computed. The first approximations for the distance the cylinder travels before separation begins is computed for various values of α from $-.5<\alpha<10$ and as $\alpha\to\infty$. Both the first and second approximations are obtained when V has the second form. This then gives the second approximation for $V(t)=At^\alpha$ when $\alpha\to\infty$. The second approximations for $\alpha=0$ and $\alpha=1$ have previously been computed by Goldstein and Rosenhead [Proc. Cambridge Philos. Soc. **32**, 392–401 (1936)] and by Blasius [Z. Math. Phys. **56**, 1–37 (1908)] respectively. Also an approximate method, based on the momentum equation, is developed for finding the separation distance, when separation starts at the rear stagnation point, and is applied to the case $\alpha=0$ with reasonable results. *R. C. DiPrima* (Cambridge, Mass.).

Sanyal, Lakshmi. On 'similarity' solutions of Prandtl's boundary layer equations. Proc. Nat. Inst. Sci. India. Part A. **21**, 8–14 (1955).

This paper is concerned with finding "similarity" solutions of the steady two-dimensional boundary-layer equations for an incompressible, viscous fluid. Rather than use the known velocity $U(x)$ of the potential flow outside the boundary layer as a scale factor for the longitudinal velocity the author has introduced an unknown scale factor, $h(x)$, of the dimensions of velocity. Also a scale factor, $g(x)$, for the distance measured perpendicular to the boundary is introduced. These functions are then determined by requiring that the boundary-layer equations reduce to an ordinary differential equation. Knowing g and h, one can deduce U. As particular cases of the general solution one obtains the well known cases in which $U(x)\sim x^m$, $U(x)\sim\exp(px)$, $U(x)=$const., and $U(x)=0$ (when a two-dimensional jet issues from a thin slit into an incompressible fluid at rest). *R. C. DiPrima.*

Li, Ting-Yi. Simple shear flow past a flat plate in an incompressible fluid of small viscosity. J. Aero Sci. **22**, 651–652 (1955).

Viguier, Gabriel. Etude mathématique de la transition, envisagée dans le cas particulier de la plaque plane. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 622–642 (1955).

Wuest, W. Asymptotische Absaugegrenzschichten an längsangeströmten zylindrischen Körpern. Ing.-Arch. **23**, 198–208 (1955).

It is assumed that v_0, the flow rate through the surface, is independent of x, the axial coordinate. The Navier-Stokes equations are thereby simplified. They are expressed in coordinates appropriate for the cylinder cross-

section under consideration. First it is assumed that the cross-flow is a potential flow; this leads immediately to a generalization of the asymptotic boundary layer for a flat plate. Examples in this category are worked out, including stationary and rotating circular cylinders with uniform v_0, an elliptical cylinder with nonuniform v_0, and the case of two intersecting planes, for which again nonuniform v_0 is required. Secondly, it is assumed that the crossflow exhibits constant vorticity, so that far from the cylinder the fluid rotates like a solid. Two examples are treated: the circular cylinder, for which the solution turns out to be the same as without rotation of the fluid, and the elliptical cylinder, where the solution (not presented) is shown to be affected by rotation. By consideration of flows with nonuniform vorticity it is shown that the solutions found previously are non-unique. Approximate solution is proposed for case of nearly irrotational cross-flow. *W. R. Sears* (Ithaca, N.Y.).

Wundt, H. Wachstum der laminaren Grenzschicht an schräg angeströmten Zylindern bei Anfahrt aus der Ruhe. Ing.-Arch. **23**, 212–230 (1955).

For yawed infinite cylinders it is well-known that the cross-flow boundary-layer components are independent of the yaw. This holds for both steady and nonsteady flows. Here the solution of problems involving acceleration of the cylinder from rest is undertaken by the method of iteration used by Blasius [Z. Math. Phys. **56**, 1–37 (1908)] and others, which leads to a power series in time t. Some of the resulting functions can be carried over from the work of Blasius [loc. cit.], Goldstein and Rosenhead [Proc. Cambridge Philos. Soc. **32**, 392–401 (1936)], and Görtler [Ing.-Arch. **14**, 286–305 (1944); MR **6**, 193]. Some are recomputed and new functions computed and tabulated here. Application of this method is made here to the case of impulsive motion of a circular cylinder. Numerical results are presented as curves of constant slope, relative to the normal cross-section, of streamlines, in the boundary layer, for various instants (actually, distances) after the start of the motion. Results of this kind are presented for values of time (distance) long enough to include separation of the cross flow and some growth of the area of separation. *W. R. Sears.*

★ **Beránek, Jiří. Theorie turbulentního proudění tekutin. [Theory of turbulent flow of fluids.]** Nakladatelství Československé Akademie Věd, Praha, 1954. 92 pp. 20 kčs.

Chapter headings: 1) General properties of turbulent flow. 2) Elements of the theory of viscous flow. 3) Turbulent friction. 4) Statistical methods. 5) Kolmogorov's theory of locally isotropic turbulence. 6) The spectrum of turbulence. 7) Applications of the statistical theory of turbulence in hydraulics. 8) Supplements.

Kampé de Fériet, J. Introduction to the statistical theory of turbulence. I, II, III, IV. J. Soc. Indust. Appl. Math. **2**, 1–9, 143–174, 244–271 (1954); **3**, 90–117 (1955).

These lectures appeared earlier as mineographed notes [University of Maryland, 1951; MR **13**, 398].

Barenblatt, G. I. On the motion of suspended particles in a turbulent flow taking up a half-space or a plane open channel of finite depth. Prikl. Mat. Meh. **19**, 61–88 (1955). (Russian)

Cet article utilise les résultats obtenus par l'auteur dans une recherche précédente [Prikl. Mat. Meh. **17**, 261–274 (1953); MR **15**, 478]. Le mouvement dépend d'une certaine grandeur sans dimensions K analogue au nombre bien connu de Richardson. Pour K très petit par rapport à l'unité la théorie de diffusion représente assez bien le mouvement. L'auteur étudie le cas où K est comparable à l'unité. Le caractère du mouvement dépend d'un paramètre sans dimensions $w=a/\varkappa v$ où $\varkappa$ est la constante de Kármán, v est la vitesse du frottement, et a est la grandeur hydraulique des particules. Les résultats expérimentaux de V. Vanoni [Proc. Amer. Soc. Civil Engrs. **70**, 793–828 (1944)] paraissent confirmer assez bien les résultats de l'auteur. *M. Kiveliovitch* (Paris).

Velikanov, M. A. On G. I. Barenblatt's paper on suspended sediment. Vestnik Moskov. Univ. **9**, no. 12, 27–31 (1954). (Russian)

Une critique d'un travail de M. Barenblatt sur les particules suspendues dans un fluide turbulent [Prikl. Mat. Meh. **17**, 261–274 (1953); MR **15**, 478]. *M. Kiveliovitch.*

Velikanov, M. A. On A. N. Kolmogorov's paper, "On a new variant of M. A. Velikanov's gravitational theory of motion of suspended sediment". Vestnik Moskov. Univ. **9**, no. 8, 39–43 (1954). (Russian)

Une réponse à une critique de M. Kolmogoroff [même Vestnik **9**, no. 3, 41–45 (1954); MR **16**, 421].

M. Kiveliovitch (Paris).

Castoldi, Luigi. Di una generale alternativa cinematica, per velocità subsoniche e ipersoniche, nel moto stazionario di un fluido comprimibile perfetto in regime adiabatico. Rend. Sem. Fac. Sci. Univ. Cagliari **24**, 46–50 (1954).

The differential relations governing the steady quasi-one-dimensional isentropic flow of a compressible fluid through a channel of slowly varying cross-section are generalized to steady isentropic three-dimensional flows. The channel direction, x, is replaced by the streamline direction, s, and the term involving the channel cross-sectional area $(1/A)(dA/dx)$, is replaced by the divergence of the unit vector tangent to the streamlines. It should be noted that this result is to be expected if the three-dimensional stream filaments are each considered as separate channels. *P. Chiarulli* (Washington, D.C.).

Morgan, A. J. A. On a class of two-dimensional channel flows with a straight sonic line. J. Aero. Sci. **22**, 573–575 (1955).

Germain, P. Maximum theorems and reflections of simple waves. NACA Tech. Note no. **3299**, 22 pp. (1955).

L'auteur considère les écoulements plans permanents d'un fluide compressible, non visqueux et non conducteur; il étudie les réflexions d'une détente par ondes simples concourantes, premièrement sur une paroi rectiligne, deuxièmement sur une ligne de courant le long de laquelle la pression est constante. La fonction de courant, déterminée dans le plan de l'hodographe au moyen de la transformation de Fourier, est reliée à la fonction de Riemann de l'équation qu'elle vérifie. Un théorème de majoration dû à L. Bers [NACA Tech. Note no. **2058** (1950); MR **12**, 61] permet de montrer que, dans le premier cas, les ondes simples concourantes se réfléchissent suivant une détente et qu'aucune ligne limite n'apparaît dans l'écoulement réfléchi. Dans le second cas au contraire, une ligne limite apparaît dans l'écoulement

réfléchi et on obtient une majorante pour la vitesse à laquelle a lieu cette apparition. Dans la dernière section, ces résultats sont reliés à la théorie mathématique des fonctions définies positives. *H. Cabannes* (Marseille).

Teodorescu, N. L'onde de choc dans la théorie invariante de la propagation des ondes. Acad. Repub. Pop. Române. Bul. Ştti. Secţ. Ştti. Mat. Fiz. **5**, 19–39 (1953). (Romanian. Russian and French summaries)

Let $u(M)$ satisfy the linear partial differential equation

$$(*) \qquad a^{ij}\partial^2 u/\partial x^i \partial x^j + b^i \partial u/\partial x^i + cu = f$$

of hyperbolic type in n-dimensional space-time V_n. If on some $(n-1)$-dimensional hypersurface S_{n-1} of V_n a first partial derivative of u in an exterior direction has a jump discontinuity while the first partial derivatives in interior directions are continuous, then S_{n-1} is called a shock wave. Let α be the jump of u and let the jump of the exterior first partial derivative be characterized by λ. If the values of α on S_{n-1} and of λ on an $(n-2)$-dimensional subspace S_{n-2} of S_{n-1} determine λ on S_{n-1}, the shock is said to be propagable. A shock is propagable if and only if S_{n-2} is space-like and S_{n-1} is a characteristic hypersurface of (*). The author derives a linear ordinary differential equation governing the variation of λ on the bicharacteristics of S_{n-1} and calls attention to a term omitted in a similar discussion in J. Hadamard's Leçons sur la propagation des ondes. [Hermann, Paris, 1903]. Finally the propagation of the wave S_{n-2} in S_{n-1} is interpreted in terms of the concept of corpuscle, a united multiplicity of $(n-2)$-dimensional contact elements with centers on S_{n-2}, previously introduced by the author in an earlier (unavailable) paper [An. Acad. Repub. Pop. Române Sect. Sti. Mat. Fiz. Chim. Ser A. 3, no. 27 (1950)]. *J. H. Giese.*

Hain, K., und v. Hoerner, S. Instationäre starke Stossfronten. Z. Naturf. **9a**, 993–1004 (1954).

The problem is considered of a strong, plane, unsteady shock front which propagates into a region of gas at rest with constant density. A numerical method based on the method of characteristics is given for the development with time of an arbitrary given initial distribution of velocity, density and pressure behind the front. Some examples have been calculated. They show that the development of the front and the following motion, for the author's various initial distributions (so chosen that no secondary shock-waves form behind the front) is, with increasing time, eventually represented closely by one and the same 'similarity solution' [C. F. v. Weizsäcker, Z. Naturf. **9a** 269–275, (1954)] of the equations of unsteady anisentropic flow; the appropriate solution turns out to be the singular solution of the family, found by W. Häfele [unpublished]. *D. C. Pack* (Glasgow).

Sakurai, Akira. On exact solution of the blast wave problem. J. Phys. Soc. Japan **10**, 827–828 (1955).

Samelson, Klaus. Überschallströmung um unter kleinem Anstellwinkel angeblasene Drehkörper mit anliegender Kopfwelle. Z. Angew. Math. Mech. **35**, 170–175 (1955). (English, French and Russian summaries)

The differential equations of supersonic flow are simplified by assuming small perturbation of axisymmetric flow, of the order of the angle of inclination of the body axis to the stream. The result is a linear system whose coefficients involve the (known) solution of the axisymmetric problem. It is shown that the attached shock wave is not only inclined, compared to its counterpart in symmetrical flow, but is also deformed, in general. No numerical results are presented here.

In a footnote added in proof, the author refers to a similar calculation by Ferri [NACA Tech. Note no. **1809** (1949); NACA Rep. no. **1044** (1951); MR **10**, 491; **14**, 331]. He also refers to Ferri's discussion of singularities of the entropy distribution [NACA Tech. Note no. **2236** (1950); NACA Rep. no. **1045** (1951); MR **12**, 875; **14**, 331]. This phenomenon and its practical effects, however, have been discussed more recently by Ferri, Ness, and Kaplita [J. Aero. Sci. **20**, 563–571 (1953)]. *W. R. Sears.*

Van Dyke, Milton D. Supersonic flow past oscillating airfoils including nonlinear thickness effects. NACA Rep. no. **1183** (1954), ii+17 pp. (1955).

Supersedes NACA Tech. Note **2982** (1953); MR **16**, 419.

Carafoli, Elie, et Horovitz, Beatrice. L'influence des disques axiaux sur l'écoulement supersonique autour des ailes angulaires coniques, à épaisseur et incidence variables. Com. Acad. R. P. Romîne **4**, 271–283 (1954). (Romanian. Russian and French summaries)

★ **Fung, Y. C. An introduction to the theory of aeroelasticity.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1955. xi+490 pp. $10.50.

This book, based on a course given at the California Institute of Technology since 1948, admirably fills the need for a comprehensive text in the rapidly expanding field of aeroelasticity. The general level, possibly excepting the details of the aerodynamic derivations of Chs. 12–14, is suitable for first-year graduate students, while the collection of material from diverse sources and the extensive bibliography (830 entries) render it a valuable reference for the practicing aeroelastician. No attempt is made to give extensive coverage to the calculation of natural vibration modes, since this topic is adequately covered elsewhere.

Ch. 1 deals with structural preliminaries, influence functions and coefficients, elementary aerodynamics, Lagrange's equations, and coupled oscillations. Ch. 2 briefly surveys flutter of non-streamlined structures such as suspension bridges. Ch. 3 uses the elementary problem of wing torsional divergence to introduce the Rayleigh-Ritz solution via assumed modes and matrix formulation via influence coefficients; a brief exposition of matrix fundamentals is included. Ch. 4 covers steady-state aeroelastic problems in general, including control-surface effectiveness and reversal, lift distribution over wings of arbitrary planforms (emphasizing the use of influence coefficients and matrices), static stability, and twisting of propeller blades. Ch. 5 introduces the elements of classical wing flutter. Ch. 6 deals with the flutter analysis of straight wings using a two-dimensional section — first on the assumption of quasi-steady flow to bring out the main features and then on the basis of unsteady-flow theory. Ch. 7 discusses briefly (as befits the author's emphasis on fundamentals) the extension to three-dimensional analysis, emphasizing the importance of practical experience. Ch. 8 takes up transient loading and develops the statistical approach to the gust-loading problem. Ch. 9 gives an introductory discussion of buffeting and stall flutter. Ch. 10 introduces the Laplace transform and its application to response and stability

problems, including the Cauchy-Nyquist criterion. Ch. 11 takes up the general formulation of aeroelastic problems with the aid of block diagrams and functional operators. Ch. 12 presents the basic equations governing ideal fluid flow. Ch. 13 derives the forces acting on a two-dimensional oscillating airfoil in an incompressible flow and discusses available, numerical results. Ch. 14 derives Possio's integral equation for subsonic compressible flow and discusses methods of solution, solves the supersonic problem explicitly, and reviews tabulated results. Ch. 15 covers briefly more general motions, three-dimensional effects, and experimental results.

The principal shortcomings of this book (viewed as a text) are, in the opinion of the reviewer: (a) absence of problems to be worked out by the student; (b) absence of discussion of first-order (as opposed to quasi-steady) frequency approximations; and (c) the omission of a discussion of aerodynamic influence coefficients (paralleling the emphasis on structural coefficients), which probably will be used with increasing frequency in digital analysis of nonsimple wings. These objections, however, are small in comparison with the book's overall advantages — viz., wide range of subject material, excellent balance between physical discussion and mathematical analysis, and general emphasis on alternative methods of attack. *J. W. Miles* (Los Angeles, Calif.).

Morgan, G. W., and Ferrante, W. R. Wave propagation in elastic tubes filled with streaming liquid. J. Acoust. Soc. Amer. **27**, 715–725 (1955).

The problém in question has an important application to the flow of blood under the oscillating pumping action of the heart. This paper is an abbreviation of an elaborate investigation, previously issued in two Office of Naval Research reports by Brown University, apparently being the most complete discussion of this subject to date. The fluid is considered incompressible but slightly viscous; the hydrodynamic analysis accordingly involves boundary-layer theory. The tube is considered elastic; the effect of Poisson's ratio was included and found to be significant. The tube is considered thin, but the effect of its inertia is considered. The limits of the results as the viscosity tends to zero are compared with those arising from inviscid theory. In the two cases the phase velocities agree only to the lowest orders of approximation. The analysis of the oscillations involves a perturbation technique assuming small amplitudes so that linearization is justified. The computations are somewhat involved. *E. Pinney* (Berkeley, Calif.).

Hooyman, G. J., Mazur, P., and de Groot, S. R. Coefficients of viscosity for a fluid in a magnetic field or in a rotating system. Physica **21**, 355–359 (1955).

The linear coefficients (generalized viscosities) relating the viscous pressure to the velocity gradient are described and classified for an isotropic fluid in a magnetic field or, equivalently, in a rotating coordinate system. It is shown that there are at most eight non-vanishing coefficients of which only seven are independent; there is one Onsager relation. Five of these are the usual coefficients of viscosity, another is the coefficient of bulk viscosity and the remaining one is a cross-effect between the ordinary and bulk viscosities. For vanishing magnetic and an isotropic medium, there are only two coefficients, one of which is the bulk viscosity.

G. Newell (Providence, R.I.).

Proudman, J. The propagation of tide and surge in an estuary. Proc. Roy. Soc. London. Ser. A. **231**, 8–24 (1955).

The interaction of a tide and a surge along an estuary is studied and some particular examples are worked out in detail. The author includes the effects of shallow water and friction, the latter being assumed proportional to the square of the speed of the current. The method used is an extension of one developed by Airy [Tides and waves, Encyclopaedia Metropolitana, London, 1842], and is essentially an iterative process, so that some account is taken of second-order terms in the equation of motion. The cases examined include those of a progressive wave in a contracting estuary, two waves in an estuary of uniform section and standing oscillations in an estuary of uniform section. In the last case the effect of the shape of the time graph of the surge on the motion is discussed, while for a progressive wave in an estuary of uniform section it is found that the maxima of surges up the estuary are less when they occur at the time of tidal high water than when they occur at the time of tidal low water. *M. H. Rogers* (Urbana, Ill.).

Kibel', I. A. Spatial problem of air flow about an unevenness in the surface of the earth. Dokl. Akad. Nauk SSSR (N.S.) **100**, 247–250 (1955). (Russian)

L'auteur se propose de donner une solution du problème dans l'espace qui n'a presque pas été étudié. Pour pouvoir arriver à une solution approximative l'auteur introduit un certain nombre de simplifications, à savoir: a) on néglige les phénomènes dans la couche limite au voisinage de la Terre; b) l'atmosphère est considérée comme un fluide idéal en mouvement stationnaire et adiabatique; et c) on se borne seulement au problème linéaire.

En introduisant ces simplifications le problème revient à trouver l'expression de la vitesse verticale w qui satisfait à une équation aux dérivées partielles du 4-ème ordre en x, y et z. Pour résoudre cette équation l'auteur développe w en série de Fourier en fonction de l'altitude. On obtient de cette façon différentes solutions pour différents types d'obstacles. L'auteur se propose dans un travail ultérieur de comparer les résultats théoriques aux observations. *M. Kiveliovitch* (Paris).

Smolyakov, P. T. On the plane problem of steady motion in the atmosphere with complete or partial independence of viscosity. Izv. Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk. **3**, 54–58 (1953). (Russian)

L'auteur étudie en détails les différents types de mouvement surtout dans le cas barotrope en négligeant la viscosité. Vers la fin de son étude il examine le cas où l'on néglige seulement la force tangentielle de la viscosité. Les résultats obtenus sont assez conformes à la pratique de l'analyse synoptique. *M. Kiveliovitch.*

Štokman, V. B. On the computation of the "lateral" friction in the dynamics of ocean flows (criticism of Hidaka's results). Dokl. Akad. Nauk SSSR (N.S.) **88**, 795–798 (1953). (Russian)

L'auteur montre en se basant sur ses recherches antérieures que l'étude de la circulation horizontale provoquée par le vent dans un océan non homogène nécessite la résolution d'une équation aux dérivées partielles de la forme

$$(1)\qquad \frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=-\frac{\operatorname{rot}\tau(x,y)}{A_1}$$

avec les conditions initiales $\psi=0$, $\partial\psi/\partial n=0$ le long du contour L où n est la normale au contour. M. Hidaka résout cette équation en négligeant le terme $2\partial^4\psi/\partial x^2\partial y^2$ et en ajoutant un terme provenant de la fonction de Coriolis. L'auteur montre en introduisant les coordonnées polaires que, tandis que la solution de (1) peut admettre une symétrie sphérique, les solutions obtenues par M. Hidaka n'admettent pas des solutions pareilles.

M. Kiveliovitch (Paris).

Štokman, V. B. Some questions of the dynamics of ocean flows. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1953, no. 1, 69–77 (1953). (Russian)

L'auteur montre que la variation en latitude du paramètre de Coriolis pour les mers de petites dimensions, est extrêmement petite, ce qui permet de calculer grossièrement l'épaisseur de la couche barotrope à l'intérieur de laquelle se produit la circulation provoquée par le vent.

M. Kiveliovitch (Paris).

Fofonoff, N. P. Steady flow in a frictionless homogeneous ocean. J. Marine Res. 13, 254–262 (1954).

Les modèles théoriques proposés les dernières années par Stommel, Munk et autres sont limités dans leurs applications par suite de la nécessité de linéariser le problème, pour éviter les complexités analytiques. C'est pourquoi l'auteur se propose d'étudier le problème dans un cas très simple d'un océan limité, de densité constante, d'un mouvement horizontal et uniforme, les effets de la thermoconvection, de la compressibilité et du frottement étant négligeables.

En utilisant le théorème de la constance de la composante du tourbillon absolu le long d'une ligne de courant: $f+\zeta=$Cte, où f est le paramètre de Coriolis et ζ la composante verticale du tourbillon relatif, on trouve assez aisément que les courants d'Est sont toujours très minces, de grande vitesse, et le tourbillon relatif a une très forte intensité. La théorie est aussi applicable à un océan composé de deux couches de densités différentes, à condition que la divergence horizontale soit négligeable.

M. Kiveliovitch (Paris).

Weibel, E. S. On Webster's horn equation. J. Acoust. Soc. Amer. 27, 726–727 (1955).

In Webster's horn theory the propagation of sound through a tube is simplified so as to involve only plane wave fronts. In the author's theory, the wave fronts are approximated by the surfaces of constant stream potential. From this stream potential, the volume displacement, as dependent on the point under consideration and on time, is calculated by application of Hamilton's principle of least action. The final equations for the pressure and the velocity have the same form as Webster's horn equation, the coefficients, however, being defined differently. *M. J. O. Strutt* (Zurich).

Robey, Donald H. On the radiation impedance of an array of finite cylinders. J. Acoust. Soc. Amer. 27, 706–710 (1955).

The author considers an array of n radiating sources and sets out to derive a formula for the radiation impedance of the qth source. The impedance of each source is treated as consisting of a self-impedance, which is the impedance that the source would have if the other sources were absent, plus $n-1$ mutual impedances. The sources are planted in a rigid baffle. In solving this problem, a Green's function with respect to the baffle is introduced. From this function, the above impedances may be calculated by integration. As a first case, a source having the shape of a circular cylinder is considered, being part of an infinitely long cylindrical baffle. Here, the Green's function is found in the form of an infinite integral expression involving Hankel and exponential functions in the integrand. Then the problem is extended to n circular-cylindrical sources of equal mutual spacing, being also part of an infinitely long cylindrical baffle. The results are evaluated and given as graphs. *M. J. O. Strutt.*

Rževkin, S. N. Sound radiators with traveling waves. Vestnik Moskov. Univ. 9, no. 8, 3–17 (1954). (Russian)

The author examines the velocity potential of a sound wave excited by a rotating body. The solution is obtained in terms of a prescribed velocity on the surface of a sphere:

$$u=U_m \sin^m \theta\, e^{j(\omega t-m\varphi)}.$$

Such a velocity distribution can be obtained by rotating a sphere with m ridges along meridians. The radiation field is examined in detail. Similar results are obtained in the case of a cylindrical radiator. The similarity between the latter and a plane surface wave radiator using a periodic surface structure is discussed. Some experimental results are given. *J. Shmoys* (Brooklyn, N.Y.).

Elasticity, Plasticity

Fokker, A. D. Nomenclature of strain parameters. Physica 21, 575–578 (1955).

The author defines three strain invariants which are very complicated functions of displacement gradients. One is a logarithmic measure of volume change, the other two measuring changes of shape. *J. L. Ericksen.*

Kröner, Ekkehart. Die Spannungsfunktionen der dreidimensionalen anisotropen Elastizitätstheorie. Z. Physik 140, 386–398 (1955).

In a previous paper [Z. Physik 139, 175–188 (1954)], the author discussed a general solution of the equations of linear elasticity for isotropic materials. This paper extends the solution and discussion to anisotropic materials. The author claims the solution is particularly useful in treating "innere Spannung," which is a stress such that the stress vector vanishes on the boundary of the region considered and the associated "strain" fails to satisfy the usual compatibility conditions. In an example given, the latter fail to hold only along a line. If they fail to hold in a region, "strain" does not have its customary kinematical significance, so it is difficult to interpret his analysis pertaining to this situation. *J. L. Ericksen.*

Mossakovskiĭ, V. I. On modelling of the first fundamental problem of the plane theory of elasticity for multiply connected regions. Prikl. Mat. Meh. 19, 383 (1955). (Russian)

Colombo, Giuseppe. Maggiorazioni delle componenti di stress nel problema di de Saint-Venant. Rend. Sem. Mat. Univ. Padova 24, 70–83 (1955).

The author considers the St. Venant torsion and flexure problems for an elastic, isotropic, cylindrical body whose cross-section C is a simply connected region. Using Cimmino's formulae for the Dirichlet problem, the

majorants of the stress components are determined. In the case when the section is circular the majorizing values coincide with those of the moduli of the stress components. Let $E_1, \cdots, E_n$, be nonoverlapping ellipses inside C with semi-axes a_r, b_r and boundary Σ_r for E_r. Let the torsion stress function be written in E_r as $\Phi=u_r+v_r$, where $\Delta_2 u_r=-K$, $\Delta_2 v_r=0$ inside E_r and $u_r=0$, $v_r=\Phi$ on the contour Σ_r. The author obtains a lower bound for the torsional rigidity M_t/K:

$$0\leq K\leq \frac{2M_t}{\pi}\left(\sum_r \frac{a_r^3 b_r^3}{a_r^2+b_r^2}\right)^{-1}.$$

Supposing $\Phi=-\frac{1}{4}K(x^2+y^2)+\psi$, the torsion problem reduces to $\Delta_2\psi=0$ (inside C), $\psi=\frac{1}{4}K(x^2+y^2)$ (on Σ); the shearing stresses satisfy the inequalities

$$|\tau_{zx}|\leq\left|\frac{d\Phi}{dn}\right|_{\max},\quad |\tau_{yz}|\leq\left|\frac{d\Phi}{dn}\right|_{\max}.$$

These inequalities become sharper as the section approaches a circle. The majorants have geometrical character only and do not depend upon the material. The majorants for the shearing stresses in the flexure problem are obtained in a similar way. It is shown also that the absolute value d of the distance of the shear centre from the neutral axis satisfies $0\leq d\leq \bar{M}_z/\delta$.

D. Rašković (Belgrade).

Mișicu, M. Extension du problème de la torsion. Com. Acad. R. P. Romîne **4**, 509–512 (1954). (Romanian. Russian and French summaries)

Dans cet article on présente l'extension du problème de la torsion des barres à section variable. En partant des équations de Lamé (3), on déduit les conditions de compatibilité (4), (5) et les équations de contour (7). On déduit que la déformation a un caractère incompressible. Le problème généralisé de la torsion est réduit à la solution d'un problème du type Neumann, en utilisant les formules d'itération (12) et (13). *Resumé de l'auteur.*

Kostandyan, B. A. On torsion of a shaft with an annular groove of rectangular form. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **7**, no. 4, 23–53 (1954). (Russian. Armenian summary)

A solution of the torsion problem for a solid cylindrical shaft consisting of three circular cylinders of different radii, twisted by axially symmetric tractions applied to the lateral surface of the shaft is given. The solution is obtained in the series of Bessel's functions, the coefficients in which can be determined from a completely regular system of linear algebraic equations. Explicit formulas for the determination of stresses are recorded.

I. S. Sokolnikoff (Los Angeles, Calif.).

Friedmann, N. E., and Rosenthal, D. Solution of combined bending and torsion problems by means of the electrical conducting sheet analogy. J. Aero. Sci. **22**, 571–572 (1955).

Vekua, I. N. On the solution of boundary problems of the theory of shells. Soobšč. Akad. Nauk Gruzin. SSR **15**, 3–6 (1954). (Russian)

The author outlines an existence proof for boundary-value problems of shell theory by reduction to the corresponding problems of the plane theory of elasticity. This process leads to Fredholm integral equations from which approximate solutions may be constructed.

J. R. M. Radok (Providence, R.I.).

Sapondžyan, O. M. Bending of an elliptic plate. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **7**, no. 5, 19–43 (1954). (Russian. Armenian summary)

The problems concerned with small transverse deflections of a clamped isotropic elliptic plate are solved for the following types of loading: a) A uniform load distributed over the area of an ellipse confocal with the given ellipse bounding the plate; b) a uniform load distributed over the area of a circle whose center coincides with the center of the ellipse; c) a uniform line load distributed at only one of the principal diameters of the ellipse. The method of solution is that used by N. I. Muskhelishvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow. Leningrad, 1949 (1953), § 64; MR **11**, 626; **15**, 370] to solve the plane problems of elasticity for an elliptical region. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sapondžyan, O. M. Bending of a semi-elliptic plate. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **7** (1954), no. 6, 27–34 (1955). (Russian. Armenian summary)

This is an improvement on Galerkin's solution of the problem of small transverse deflections of a uniformly loaded semi-elliptic plate clamped along its curvilinear boundary and simply supported along the rectilinear boundary [Elastic thin plates, Gosstrolizdat, Leningrad-Moscow, 1933, pp. 340–343]. Since the deflections of a clamped elliptic plate carrying a uniform load p above its principal diameter and $-p$ below the diameter are the same as in the problem treated by Galerkin, the method of solution employed by the author in the paper reviewed above is applicable. The formulas deduced by him are more tractable than Galerkin's. *I. S. Sokolnikoff.*

Mossakowski, Jerzy. Singular solutions for anisotropic plates. Arch. Mech. Stos. **7**, 97–110 (1955). (Polish. Russian and English summaries)

The author makes use of some particular integrals of the differential equation for small transverse deflections of thin anisotropic elastic plates to obtain the deflections and moments in an infinite plate under a concentrated load. The work is based on a representation of deflections by means of functions of certain complex variables determined by the anisotropy of the medium [see S. G. Lehnickiĭ, Anisotropic plates, OGIZ, Moscow-Leningrad, 1947; MR **10**, 415]. *I. S. Sokolnikoff.*

Šerman, D. I. On the bending of a circular plate partly clamped and partly supported on the contour. Dokl. Akad. Nauk SSSR (N.S.) **101**, 623–626 (1955). (Russian)

This paper contains an elegant solution of the problem of tranverse deflection of a uniformly loaded circular plate one-half of whose boundary (γ_1) is clamped and the other half (γ_2) is simply supported. The determination of deflection w is reduced to the search for one harmonic function v satisfying on γ_1 and γ_2 mixed boundary conditions. If $v=\Re[\varphi(z)]$, where $\varphi(z)$ is analytic in the region occupied by the plate, these boundary conditions can be expressed in terms of φ. By a familiar device, a singular integral equation of Cauchy's type is deduced from the boundary conditions. Finally, by introducing the function $\varphi^*(z)=\varphi(z)-a_0$, which reduces to zero for $z=0$, a singular equation is obtained for φ^*. This equation is solved by expanding φ^* in a power series. If γ_2 is not a semicircle,

a bilinear transformation can be employed to transform it into a semicircle. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Buckens, F. Relations caractéristiques entre les valeurs critiques des forces radiales agissant sur les bords intérieurs et extérieurs d'une plaque annulaire. Bull. Soc. Franç. Méc. **5**, no. 15, 33–39 (1955).

Après un rappel de certaines propriétés des fonctions qui relient les valeurs caractéristiques dans des problèmes comportant plusieurs paramètres, l'auteur détermine la relation quasi-linéaire qui existe entre les valeurs critiques des efforts comprimant une plaque annulaire dans son plan. Les solutions sont limitées au flambage à symétrie circulaire. L'auteur considère successivement une plaque encastrée sur les bords extérieur et intérieur, et sur le seul bord intérieur. Des exemples numériques permettent de mettre en évidence l'utilisation de ces relations caractéristiques en vue de résoudre des problèmes concrets. Dans certains cas, la très faible concavité de la première courbe caractéristique peut servir à déterminer une approximation par défaut des valeurs propres cherchées.

R. Gran Olsson (Trondheim).

Bonneau, Louis. Problèmes d'élasticité à trois dimensions. L'équation de Lagrange et les plaques rectangulaires. Ann. Ponts Chaussées **125**, 403–437 (1955).

The author uses 3-dimensional elasticity to investigate some problems in flat-plate theory. Taking the middle surface of the plate as $z=0$, he calculates σ_{zz} on the surface of the plate and w in the middle surface, both when the plate is supported by staunchions placed at the vertices of a rectangular network and when the support is provided by a rectangular network of bars. Unfortunately he does not compare his numerical results with those obtainable from the equation of neutral equilibrium $\nabla^4 w+Cp=0$. *D. R. Bland* (London).

Trenin, S. I. Aufbau einer Methode zur Lösung einer Reihe von axialsymmetrischen Problemen der Elastizitätstheorie. Vestnik Moskov. Univ. **7**, no. 6, 3–14 (1952). (Russian)

Verf. wählt die vier auf Zylinderkoordinaten bezogenen Spannungskomponenten des achsensymmetrischen Zustandes in der Form $S^j=S_0^j+S_k^j$ $(j=1, 2, 3, 4)$, worin die Anteile S_0^j den Differentialgleichungen des Gleichgewichtes und den vorgegebenen Belastungsbedingungen an der Oberfläche, die Anteile S_k^j dagegen wohl den ersteren genügen sollen, dabei aber verschwindende Oberflächenspannungen zu liefern haben. Zwei Spannungsfunktionen, deren Form die Erfüllung der Gleichgewichtsrelationen sicherstellt, dienen dann zum Aufbau eines Satzes von Funktionen S_0^j, S_{kn}^j $(j=1,2,3,4; n=1,2,3,\cdots)$ die dem jeweiligen Problem anzupassen sind. Die Parameter A_n^j des endgültigen Ansatzes $S^j=S_0^j+\sum_n A_n^j S_{kn}^j$ ergeben sich dann nach der Variationsmethode von Castigliano, womit auch den Bedingungen der Kompatibilität entsprochen werden kann. Als Beispiel wird in erster und zweiter Näherung ($n=1$ bzw. $n=1, 2$) die Spannungsverteilung in einem Kreiszylinder untersucht, der auf seinen Basisflächen einer glockenförmig verteilten Normalbelastung unterworfen ist, während seine Mantelfläche spannungsfrei bleibt. *S. Woinowsky-Krieger* (Zbl **47**, 424).

Abramyan, B. L. On the problem of axially symmetric deformation of a circular cylinder. Akad. Nauk Armyan. SSR. Dokl. **19**, 3–12 (1954). (Russian. Armenian summary)

An isotropic circular cylinder is undergoing infinitesimal elastic deformation under given axisymmetrical distributed loading on the curved and end surfaces. By expansion of the loading functions in series, systems of linear equations in infinitely many unknowns are set up. No problems are solved. *L. M. Milne-Thomson.*

Rostovcev, N. A. On certain cases of the contact problem. Ukrain. Mat. Ž. **6**, 326–332 (1954). (Russian)

The author extends his own method of solution [Prikl. Mat. Meh. **17**, 99–106 (1953); MR **14**, 1146] to cover the axisymmetric and the plane problem with two regions of contact. *J. R. M. Radok* (Providence, R.I.).

Albrecht, F. L'équilibre élastique des cristaux du système cubique. Com. Acad. R. P. Române **1**, 403–408 (1951). (Romanian. Russian and French summaries)

For cubic crystals, the author gives a derivation of the displacement equations and stress equations corresponding to the Beltrami-Mitchell equations for isotropic materials, assuming linear elasticity. He writes down some solutions for uniformly varying stress.

J. L. Ericksen (Washington, D.C.).

Metelicyn, I. I. On elastic impact. Ukrain. Mat. Ž. **6**, 147–175 (1954). (Russian)

The author discusses in great detail previous work on the problem of impact and, in particular, the discrepancies between the results, due to Hertz and St. Venant. In the light of experimental evidence, both these theories are often unsatisfactory, because they do not adequately allow for the kinematic conditions which take place at the instant of impact. The author develops a new approach, based on impulses, and introduces the concept of loss of kinetic energy. He also suggests that, depending on the material of the bodies, the description of the later stages of the process may not be justifiably based on the classical theory of elasticity, since the occurring strain rates may, for example, lead to plastic deformations. In defense of his approach, the author refers to the invariably occurring changes in temperature which explain his idea of loss of kinetic energy.

Comparison of results obtained in this paper with those from St. Venant's theory are given for the cases of impact of a rod on a rigid wall, of a rod on another rod with one fixed end and of two free rods. It is suggested that the amount of dissipated energy would have to be determined by experiment which might go some way towards explaining the mechanism at the time of impact.

J. R. M. Radok (Providence, R.I.).

Bondar, M. G. On dynamical stability of systems of rods. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 351–355 (1954). (Ukrainian. Russian summary)

In this paper the integro-differential equations of the dynamical stability of rod systems are given. The connection between the differential and integro-differential equations of dynamical stability is pointed out. A new proof is given to the theorem on separation of variables for the equations of dynamical stability taking account of forces of resistance. *Author's summary.*

Grigolyuk, È. I. Nonlinear oscillations and stability of sloping rods and shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 3, 33–68 (1955). (Russian)

The author analyzes vibrations of curved rods and axisymmetrical spherical and conical sloping shells at various loads and boundary conditions. Every case

represents a different boundary-value problem; therefore the author solves these boundary-value problems using different methods. For curved rods with hinged ends he uses a Fourier method, for curved rods with built-in ends an energy method, for shells Bubnov's method. Solutions for straight rods and circular plates are obtained as special cases. Bubnov's method, very briefly described, consists of orthogonalizing the integro-differential equation of the deflection surface either with the assumed expression for the deflection, or with its first derivative. The author gives no reference to a detailed exposition of Bubnov's method.

In order to make his solutions suitable for numerical computation the author includes tables with values of various coefficients and graphs. The reviewer tried to use the author's method to compute vibrations of a circular plate and found it complicated and inconvenient. There are printing errors on page 65. In the 8th line from the bottom instead of $f(W^0)$ read $f(W^0)=0$; in the 9th line from the bottom the subscripts "0" should be replaced by the same superscripts.

T. Leser (Aberdeen, Md.).

Vâlcovici, V. Le flambage d'une colonne pesante immergée dans un fluide et appuyée de tout son poids au fond. Com. Acad. R. P. Române **1**, 559–561 (1951). (Romanian. Russian and French summaries)

Vâlcovici, V. La solution graphique du problème du flambage des barres élastiques minces, pesantes, immergées dans un fluide. Com. Acad. R. P. Române **1**, 727–730 (1951). (Romanian. Russian and French summaries)

Vâlcovici, V. La formule de flambage des colonnes pesantes, immergées dans un fluide et soumises à une forte compression. Com. Acad. R. P. Române **1**, 905–907 (1951). (Romanian. Russian and French summaries)

Vâlcovici, V. Les diagrammes de flambage dans le cas des barres élastiques, lourdes, immergées dans un fluide. La détermination de la compression de sûreté. Com. Acad. R. P. Române **2**, 185–188 (1952). (Romanian. Russian and French summaries)

Popov, S. M. Stability beyond the elastic limit of plates with reinforcing ribs. Inžen. Sb. **12**, 49–76 (1952). (Russian)

The author investigates the stability of simply supported, rectangular plates with longitudinal or transverse reinforcing ribs in uniaxial compression or under uniform shear loading, when the common material becomes plastic. Using an energy approach, he obtains an infinite system of linear equations for the Fourier coefficients of the displacement solution which he solves approximately. Results for particular cases involving one to three reinforcements are tabulated. One additional problem considered is that of a plate, simply supported along two sides and reinforced along the other two sides, under uniaxial compression. *J. R. M. Radok.*

Popov, S. M. Stability of simply supported plates beyond the elastic limit. Inžen. Sb. **9**, 65–98 (1951). (Russian)

Based on the theory of small elasto-plastic deformations, the author establishes the stability equation for rectangular plates and the corresponding energy expression. Special cases studied by energy methods comprise uniaxial and biaxial compression. The case of uniaxial compression of plates with simply supported edges at right angles to the applied load and various edge conditions along the other two sides is studied by use of the equilibrium equations. Extensive numerical results are given in the form of tables and curves.

J. R. M. Radok (Providence, R.I.).

See also: Nowinski, p. 2; Birman, p. 41; Lordkipanidze, p. 90; Critescu, p. 97; Fung, p. 101; Morgan and Ferrante, p. 102.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory, Circuits

★ **Linfoot, E. H. Recent advances in optics.** Oxford, at the Clarendon Press, 1955. x+286 pp. (8 plates). $8.00.

The author has made contributions to geometrical optics, diffraction optics and to the study of coherence in optical systems. His book can be considered as a collection of a series of monographs in special fields of optics in which he is interested.

The first section deals with the optical image starting from the characteristic function (called eikonal by Bruns and ikonal by Linfoot). The reviewer may look askance at some of the methods which the author uses in defining the best image surfaces, the Petzval surface, etc., but the author's approach is always interesting and stimulating, even if sometimes oversimplifying. Diffraction is introduced early and the circle polynomial of Zernicke and Nieboer are applied to the case of small aberrations. A section entitled images of partially coherent objects, containing miscellaneous subjects finishes the first monograph.

The second monograph deals with the Foucault (knife-edge) test, its theory, and its application to the study of aberrations of optical system with small errors.

The third monograph deals with the Schmidt correcting plate, to the theory of which the author has made various contributions.

The last chapter carries out in detail an idea by C. R. Burch. The author adds to each surface a corrector plate in the sense of the Schmidt camera so that surface + corrector plate are corrected for an object within the accuracy of Seidel aberrations. The analysis is carried through for a series of simple refracting and reflecting systems. *M. Herzberger* (Rochester, N.Y.).

Signorini, Antonio. Sopra una questione di ottica geometrica. Rend. Sem. Mat. Univ. Padova **24**, 37–44 (1955).

The author investigates different isotropic inhomogeneous media in which the light rays from a point may unite in a point. He especially investigates refractive indices of the form $n=1/wk$ and $n=1/(k+h(x^2+y^2+z^2))$, where w, k, and h are constants. *M. Herzberger.*

Dumontet, P. Sur la correspondance objet-image en optique. Opt. Acta **2**, 53–63 (1955).

The author investigatez the frequently stated opinion that an optical system corrected for spherical aberration and coma can be considered as a linear filter which has

the quality that the correspondence between objects and image is linear, and that it is possible to shift the object point in its place without changing the amplitude distribution in the image. His conclusion that this is true only for a very small displacement is in agreement with the geometric-optical interpretation of the meaning of aplanatism. The author proposes a correction in the general case which he applies to the case of image formation with partially coherent illumination.

M. Herzberger (Rochester, N.Y.).

Picht, Johannes. Über den Einfluss von Fehlern bei der Herstellung optischer Systeme auf die Bildfehler dieser Systeme. Wiss. Z. Pädagog. Hochsch. Potsdam **1**, 13–28 (1955).

The author gives formulas which investigate for a single surface the influence of a change of curvature of distance and refractive index on the position and direction of a meridional ray, as well as on the position of the astigmatic points along it. A short discussion on "decentration" concludes the paper. *M. Herzberger.*

Wachendorf, F. Die Bestimmung eines optimalen Linsensystems. Optik **12**, 329–340, 345–359 (1955).

The author discusses the problem of determining an optimal optical system according to different definitions of optimal quality and derives equations to find the neighborhood optimum for a given already well corrected system. The conditions for an optimum are given in a purely formal mathematical way (by an arbitrary explicit or implicit function of the coordinates of the image points). The author then investigates the formal mathematical procedure to obtain the optimum in each specific case. *M. Herzberger* (Rochester, N.Y.).

Laudet, Michel. Optique électronique des systèmes cylindriques présentant un plan de symétrie. I. L'approximation du premier ordre. J. Phys. Radium (8) **16**, 118–124 (1955).

Partant de l'expression classique du lagrangien des systèmes électromagnétiques à deux variables

$$F(x,y,y',z')=\sqrt{((V+\varepsilon)+\alpha(V+\varepsilon)^2)}\sqrt{(1+y'^2+z'^2)}\pm\gamma z'Ax$$

avec $\gamma=\sqrt{(|q|/2m_0)}$, $\alpha=|q|/2m_0c^2$, l'auteur détermine les trajectoires des particules dans le cas général, à partir des équations de Lagrange

$$\frac{d}{dx}\left\{\frac{\partial F}{\partial y'}\right\}-\frac{\partial F}{\partial y}=0,\quad \frac{d}{dx}\left\{\frac{\partial F}{\partial z'}\right\}=0$$

et intègre ces dernières dans le cas particulier de l'approximation du premier ordre, après avoir développé en série la fonction F suivant les puissances croissantes des variables y, y' et z'. La méthode suivie est celle proposée par E. Durand pour l'étude des systèmes de révolution [Rev. Opt. **33**, 617–629 (1954); MR **16**, 652].

L'auteur met en évidence le fait que pour des systèmes cylindriques symétriques il n'y a pas stigmatisme, mais qu'il est toujours possible de définir des plans conjugués; ils sont tels, qu'une droite inclinée dans le „plan image" correspond à un point dans le „plan objet". L'intégration des équations des trajectoires paraxiales fait apparaître trois constantes indépendantes, au lieu des deux constantes que l'on rencontre dans les systèmes de révolution qui sont stigmatiques. La considération d'objets éclairés en lumière parallèle permet de dégager la notion de „plan focal image" et celle de „droite focale image". Enfin l'auteur applique les formules générales à quelques cas particuliers. *E. Durand* (Toulouse).

Cukkerman, I. I. On electron-optical systems with a rectilinear axis not having rotational symmetry. Ž. Tehn. Fiz. **24**, 2261–2263 (1954). (Russian)

Electron-optical systems with a rectilinear axis are defined as systems of static electric and magnetic fields, such that one of the trajectories of an appropriately directed beam of charged particles is a straight line, which is the axis of the system. The equations of motion for paraxial electrons in such a system are obtained as special cases of Grinberg's general equations [Selected topics of the mathematical theory of electric and magnetic phenomena, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948, pp. 507–535]. The condition is stated which has to be imposed on the components of the field in order for the system to have a rectilinear axis. A further condition is derived which has to be satisfied in order that a beam radiating from a point on the axis be brought again to a focus. A number of examples are discussed where these conditions are satisfied. *J. E. Rosenthal.*

Wolf, E. A macroscopic theory of interference and diffraction of light from finite sources. II. Fields with a spectral range of arbitrary width. Proc. Roy. Soc. London. Ser. A. **230**, 246–265 (1955).

In part I [same Proc. **225**, 96–111 (1954); MR **16**, 200], interference and diffraction of light in stationary fields produced by finite sources that emit light within a small but finite spectral range were studied. The results obtained are inadequate in the treatment of problems where the path differences between the interfering beams are large. Part II removes this restriction, and extends the results to stationary fields of any spectral range, by means of a space-time correlation function more general than Zernike's used in part I. A rigorous formulation of the propagation law for the correlation function and of the generalized Huygens principle is given. *C. J. Bouwkamp.*

Poincelot, Paul. Sur un problème de diffraction. C. R. Acad. Sci. Paris **241**, 625–627 (1955).

A plane sinusoidal electromagnetic wave is incident normally upon a thin conducting screen in the form of a rectangle of infinite length and finite width. The electric field of the incident wave is parallel to the edges, and the net electric field is to vanish on the surface of the screen. The author expresses the current density as a series of Fourier integrals, and by contour integration obtains an infinite algebraic system for the determination of the coefficients in the series. The system is asserted to be convergent. *R. N. Goss* (San Diego, Calif.).

Berg, Eduard. Phasenverhältnisse im Beugungsschatten. Arch. Elektr. Übertragung **9**, 227–230 (1955).

The phase surfaces of a vertical magnetic dipole, in the presence of a perfectly conducting spherical earth and a homogeneous atmosphere, are calculated beyond the horizon for a dipole elevation of 100 metres and a wavelength of 6π metres. The method is borrowed from Watson, Van der Pol, and Bremmer. *C. J. Bouwkamp.*

Jones, D. S. The scattering of a scalar wave by a semi-infinite rod of circular cross section. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 499–528 (1955).

"The form of the exact solution for the scattering of a plane harmonic scalar wave by a semi-infinite circular cylindrical rod of diameter $2a$ is found when the boundary condition is $u=0$ or $\partial u/\partial\nu=0$, where u represents the scalar field and ν is the normal to the rod.

"When the angle of incidence is π, i.e. the angle between the direction of propagation of the incident wave and the normal (out of the rod) to the end is π, the average pressure amplitude on the end of the rod and the scattering coefficients are found for the boundary condition $\partial u/\partial \nu=0$. Graphs are given showing the behaviour of these quantities for the range $0\leq ka\leq 10$, where k is the wave-number. It is further shown that the average pressure amplitude on the end for other angles of incidence is approximately the product of the average pressure amplitude for an angle of incidence of π and the amplitude of the symmetric mode ($ka<3.83$) which the incident field would produce inside a hollow semi-infinite cylinder occupying the same position as the rod. When the boundary condition is $u=0$ and ka is small it is proved that the scattered field is the same as that due to a semi-infinite hollow cylinder longer by an amount $0.1a$ approximately. A similar result does not hold for the boundary condition $\partial u/\partial \nu=0$.

"The theory is extended to the case when a pressure pulse falls on a circular rod. It is found that the pressure on the end drops almost to its final value in the time taken for a wave to travel the diameter of the rod." (From the author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

Vouk, V. B. The extinction cross-section coefficient of large perfectly absorbing spherical particles. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 65–71 (1954).

The paper deals with the calculation of the extinction cross-section coefficient of large perfectly absorbing spherical particles as a function of particle size and the solid angle subtended by the measuring instrument. Physical arguments are used to obtain an approximate expression for the differential scattering cross-section in the case of small, complex susceptibility, large sphere radius r. This is then integrated over a small cone of half-angle θ_0 around the forward direction. The result is a function of $(2\pi r \sin\theta_0/\lambda)$. For small θ_0 the extinction cross-section approaches twice the geometrical cross-section. *J. Shmoys* (Brooklyn, N.Y.).

Vouk, Velimir. The theory of optical cross-sections of large perfectly absorbing particles. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 123–134 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Heyn, Eugen. Beweis der Vollständigkeitsrelation für die charakteristischen Vektorfelder der Hohlleitertheorie. Math. Nachr. **13**, 25–56 (1955).

In investigations on the propagation of electromagnetic waves along a wave guide, there occur two systems of two-dimensional vector fields $\mathbf{e}_\nu'(x, y)$, $\mathbf{e}_\mu''(x, y)$, $\mathbf{e}_\sigma'''(x,y)$ and $\mathbf{h}_\nu'(x, y)$, $\mathbf{h}_\mu''(x, y)$, $\mathbf{h}_\sigma'''(x, y)$. These fields are associated with the following eigenvalue problems, I and II, and the boundary-value problem III for a region G in the (x, y)-plane with boundary S:

I) $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}+\lambda\Phi=0$ with $\Phi=0$ on S;

II) $\dfrac{\partial^2\Psi}{\partial x^2}+\dfrac{\partial^2\Psi}{\partial y^2}+\mu\Psi=0$ with $\dfrac{\partial\Psi}{\partial n}=0$ on S;

III) $\dfrac{\partial^2\chi}{\partial x^2}+\dfrac{\partial^2\chi}{\partial y^2}=0$ with χ constant on S.

I has eigenvalues $\lambda_i=k_i'^2$ $(i=1, 2, \cdots)$ and corresponding eigenfunctions Φ_i. II has eigenvalues $\mu_i=k_i''^2$ $(i=1, 2, \cdots)$ and corresponding eigenfunctions Ψ_i; $\mu_0=0$ and $\Psi_0=$ constant, but the other μ_i are positive. III has N linearly independent solutions $\chi_i (i=0, 1, 2, \cdots, N-1)$ if G has connectivity N and χ_0 is a constant. All these functions can be taken to be real without loss of generality. The associated vector fields are

$$\mathbf{e}_i'=-\frac{1}{k_i'}\operatorname{grad}\Phi_i,\ \mathbf{e}_i''=\frac{1}{k_i''}\mathbf{k}\times\operatorname{grad}\Psi_i,\ \mathbf{e}_i'''=\operatorname{grad}\chi_i,$$

$$\mathbf{h}_i'=-\frac{1}{k_i'}\mathbf{k}\times\operatorname{grad}\Phi_i,\ \mathbf{h}_i''=-\frac{1}{k_i''}\operatorname{grad}\Psi_i,$$

$$\mathbf{h}_i'''=\mathbf{k}\times\operatorname{grad}\chi_i$$

where $\mathbf{k}$ is the unit vector perpendicular to the plane of G, where it is assumed that the systems $\{\Phi_i\}$ and $\{\Psi_i\}$ are normal and orthogonal. The $N-1$ vector fields $\mathbf{e}_\nu'''$ are linearly independent and may be assumed orthogonal, using the inner product

$$(\mathbf{f}, \mathbf{g})=\iint_G \mathbf{f}\cdot\mathbf{g}\,dx\,dy,$$

and normalized. We then have

$$(\mathbf{e}_\nu', \mathbf{e}_\mu')=\delta_{\nu\mu},\ (\mathbf{e}_\nu'', \mathbf{e}_\mu'')=\delta_{\nu\mu},\ (\mathbf{e}_\nu''', \mathbf{e}_\mu''')=\delta_{\nu\mu},$$
$$(\mathbf{e}_\nu', \mathbf{e}_\mu'')=0,\ (\mathbf{e}_\nu'', \mathbf{e}_\mu''')=0,\ (\mathbf{e}_\nu''', \mathbf{e}_\mu')=0.$$

A similar set of relations connect $\mathbf{h}_\nu'$, $\mathbf{h}_\mu''$, $\mathbf{h}_\sigma'''$.

The totality of vector fields $\mathbf{e}_\nu'$, $\mathbf{e}_\mu''$, $\mathbf{e}_\sigma'''$ then form a normalised orthogonal system in the Hilbert space formed of all vector fields $\mathbf{A}(x, y)$, with components A_x and A_y of integrable square over G, the inner product being

$$(\mathbf{A}, \mathbf{B})=\iint_G \mathbf{A}(x, y)\cdot\overline{\mathbf{B}(x, y)}\,dx\,dy.$$

The author proves that this normalised orthogonal system is complete; and similarly for $\mathbf{h}_\nu'$, $\mathbf{h}_\mu''$, $\mathbf{h}_\sigma'''$.

E. T. Copson (St. Andrews).

Grosjean, C. C. Transformation of a frequency equation in corrugated wave guide theory. Nuovo Cimento (10) **1**, 174–192 (1955).

In what follows a, b, d, D, k, β_0 are positive numbers denoting certain physical quantities. Further, $\beta_m=\beta_0+2\pi m/D$, $\chi_m^2=k^2-\beta_m^2$, $K_n^2=\alpha_n^2-k^2a^2$, $J_0(\alpha_n)=0$, and J and Y are Bessel functions. The author transforms the "frequency equation" of Walkinshaw [Proc. Phys. Soc. **61**, 245–254 (1948)]

$$(1)\quad \frac{J_0(ka)Y_0(kb)-Y_0(ka)J_0(kb)}{J_1(ka)Y_0(kb)-Y_1(ka)J_0(kb)}=$$
$$=\frac{d}{D}\sum_{m=-\infty}^{\infty}\frac{k}{\chi_m}\,\frac{J_1(\chi_m a)}{J_0(\chi_m a)}\,\frac{J_0(\beta_m d/2)\sin(\beta_m d/2)}{\beta_m d/2}$$

into the equivalent

$$\frac{J_0(kb)}{\pi k^2a^2J_0(ka)[J_1(ka)Y_0(kb)-Y_1(ka)J_0(kb)]}=$$
$$=-\sum_{n=1}^{\infty}\frac{\cosh(K_nD/a)[1-(d/2D)]-\cos\beta_0D\cosh(K_nd/2a)}{K_n^2[\cosh(K_nD/a)-\cos\beta_0D]}\times$$
$$\times I_0(K_nd/2a),$$

where I_0 is the modified Bessel function. He further considers all types of numerical approximation in relation to the evaluation of the two series by truncating them after an appropriate number of terms. [The reviewer

observes that the author's equation (1) and Walkinshaw's equation differ by a factor i in the right-hand sides, but this may be due to a misprint.]

C. J. Bouwkamp (Eindhoven).

Grosjean, C. C. On the theory of circularly symmetric TM waves in infinite irisloaded guides. Nuovo Cimento (10) **1**, 427–438 (1955).

The rigorous theory of the propagation of circularly symmetric transverse magnetic waves in corrugated infinite guides with perfectly conducting walls is worked out in detail. Assuming two arbitrary functions for the description of the axial electric field component at the corrugation mouths, consistent with Floquet's theorem of differential equations with periodic coefficients, the author obtains an infinite system of linear equations in an infinite number of unknowns pertinent to the physical problem. This system is transformed in various ways leading to new formulae, the implications of which have not yet been investigated. There is some overlap with the paper reviewed above. *C. J. Bouwkamp* (Eindhoven).

Grosjean, C. C. Mathematical transformation of a system of equations appearing in the theory of TM wave propagation in corrugated guides. Nuovo Cimento (10) **1**, 439–446 (1955).

Generalization of methods and results obtained in the two papers reviewed above.

C. J. Bouwkamp (Eindhoven).

Vanhuyse, V. J. On the (β_0, k) diagrams for circularly symmetric TM waves in infinite irisloaded waveguides. Nuovo Cimento (10) **1**, 447–452 (1955).

Using the results of the paper reviewed second above in the case of infinitely thin iris walls, the author is able to obtain the diagrams mentioned in the title.

C. J. Bouwkamp (Eindhoven).

Gaponov, A. V. On the theory of thin antennae in cavity resonators. Ž. Tehn. Fiz. **25**, 1069–1084 (1955). (Russian)

The problem of free vibrations of a slotted cavity resonator containing thin antennae is formulated as the problem of free vibrations of a conventional cavity resonator with perturbations in the bounding surface. Proceeding from a system of Hallén-like integro-differential equations containing interaction terms, the author develops formulas for the eigenfunctions of the perturbed boundary-value problem and for the Q of the resonator. The results are alleged to reduce to those of Samarskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **63**, 631–634 (1948); MR **10**, 458] for the corresponding scalar problem. They are applied to the study of various cases of excitation of a cavity resonator by thin antennae.

R. N. Goss (San Diego, Calif.).

Gaponov, A. V. Excitation of a cavity resonator by thin antennae. Ž. Tehn. Fiz. **25**, 1085–1099 (1955). (Russian)

A forcing term is added to the basic equation of the paper reviewed above. The method of perturbations is used to solve the new equation for cases of excitation by tuned or untuned antennae, by external fields or external currents. In each case a solution in which the first perturbation term only is retained is given, followed by one in which the second term also is kept.

R. N. Goss (San Diego, Calif.).

King, Ronold. Gap problem in antenna theory. J. Appl. Phys. **26**, 317–321 (1955).

The author considers anew the gap problem in the theory of the cylindrical dipole antenna. He notes that no physical antenna has a gap, in the sense that any complete study of an antenna system requires the understanding of a driving mechanism. He selects for particular study a radial transmission line as an example of a driving mechanism for a cylindrical antenna and shows that this leads to a well-defined problem. The problem can only be explicitly solved, however, if some means can be found for matching the impedances in the region of discontinuity (the gap). The author achieves this match through the use of certain simplifying assumptions which he believes to be valid for the dimensions of interest.

W. K. Saunders (Washington, D.C.).

Shmoys, J. Long-range propagation of low-frequency radio waves between the earth and the ionosphere. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-79**, i+28 pp. (1955).

The problem considered is that of the propagation of low frequency radio waves between a perfectly conducting earth and an ionosphere, where the characteristic properties vary exponentially with the height.

In terms of a single-component Hertz vector the equation to be solved is

$$\nabla^2\pi+k^2\varepsilon^*\pi=0.$$

As to the boundary conditions two cases are considered corresponding to horizontal polarization and vertical polarization. In the first case, on the surface of the earth $\pi=0$; in the second case, the normal derivative of π must vanish. At infinity the Sommerfeld condition is to be satisfied. The coefficient ε^*, related to the electric constant ε, is also different in the two cases. It is pointed out, that the error incurred by treating the earth as a plane and the ionosphere as a stratified medium is negligible.

In an ionized medium ε depends on the electron density N and the collision frequency ν. The author considers separately the cases that ν is very small and that ν is great. In the case of small values of ν and horizontal polarization the second term in the differential equation may be split up in two terms, one independent of the coordinates the other independent of k. The fact that in the case of vertical polarization this splitting up is impossible gives rises to difficulties not discussed in this report. In the case of great values of ν, and ω (the angular frequency of the field) small compared to ν, approximate solutions are obtained in terms of Bessel functions.

H. Bremekamp (Delft).

Kacenelenbaum, B. Z. Perturbation of an electromagnetic field by small deformations of a metal surface. Ž. Tehn. Fiz. **25**, 516–557 (1955). (Russian)

The author calculates the first-order effect of slight deformations of a perfectly conducting plane on the electric and magnetic fields. It is shown that these deformations can be replaced, in the first approximation, by a charge and current distribution on the original plane boundary, and the radiation field of these charges and currents is then calculated. The results are then applied to the problem of diffraction by a step whose height is small compared to wave length. A comparison is made between the present method, and that of Rayleigh, as applied to the problem of scattering by a rough surface.

J. Shmoys (Brooklyn, N.Y.).

De Schwarz, M. J. e Ventura, M. L. Tensioni e correnti in un cavo pupinizzato. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89**, 193–207 (1955).

The authors want to evaluate the current and the tension at the centres of the coils of a pupinized cable as dependent on time. Each of these quantities is given by an integral expression involving antitransformation functions of Laplace in their integrands. These functions are defined, their existence is proved and their evaluation by means of convergent infinite series is shown.

M. J. O. Strutt (Zurich).

Oblonský, Jan. Electromagnetic relay with suppressed inductive coupling between windings. Stroje na Zpracování Informací **2**, 261–270 (1954). (Czech. Russian and English summaries)

Moon, Parry, and Spencer, Domina Eberle. On electromagnetic induction. J. Franklin Inst. **260**, 213–226 (1955).

Giambiagi, J. J. Maxwell's equations for multipoles. Rev. Un. Mat. Argentina **16**, 89–90 (1955).

Nicolau, Edmond. Transcription des équations maxwelliennes à l'aide des tenseurs complexes. Com. Acad. R. P. Romîne **4**, 613–619 (1954). (Romanian. Russian and French summaries)

Mariot, Louis. Sur le champ électromagnétique singulier. C. R. Acad. Sci. Paris **241**, 175–176 (1955).

An electromagnetic field is called singular if the electromagnetic energy-tensor $\tau_{\alpha\beta}$ is of the form $\tau_{\alpha\beta}=P^2 l_\alpha l_\beta$, where P is a scalar and l_α is a null-vector. The trajectories of l_α are null geodesics, as shown by the author in an earlier paper [same C. R. **238**, 2055–2056 (1954); MR **15**, 995]. The singular field is said to be completely integrable if the field of 3-planes $l_\alpha dx^\alpha=0$ is completely integrable. The author proves the following theorem: Given on a space-like hypersurface S a singular field such that the null directions l of this field on S define, through the agency of the corresponding congruence of geodesics, a completely integrable field of 3-planes, then the corresponding unique solution of Maxwell's equations in the neighbourhood of S is a completely integrable singular field.

H. S. Ruse (Leeds).

Szymański, Z. On linear electrodynamics with higher derivatives. Bull. Acad. Polon. Sci. Cl. III. **3**, 217–222 (1955).

The author considers generalizations of classical electrodynamics in which the Lagrangian density contains derivatives of the four-vector potential, $A_\alpha(x)$, $\alpha=1,\cdots,4$, up to the nth order. Under the assumptions of 1) relativistic invariance, 2) gauge invariance, 3) linearity of the field equations, he shows that the most general field equation can be written in the form

$$\prod_{j=1}^{n-1}(\Box-\mu_j^2)A_\alpha(x)=0. \tag{1}$$

This shows that the formalism assumed can give no more general equations than (1), which was the starting point of the quantum-mechanical investigation of Pais and Uhlenbeck [Phys. Rev. (2) **79**, 145–165 (1950); MR **12**, 227].

A. S. Wightman (Princeton, N.J.).

Podolsky, Boris, and Denman, Harry. A derivation of generalized macorscopic electrodynamic equations. I. Non-relativistic. J. Math. Phys. **34**, 198–207 (1955).

A discussion is given of the formal mathematical theory underlying the derivation of the electromagnetic field equations in material media from those for free space containing charge and current distributions. Classical physics is used as a basis, the charge and current distributions being entirely those of point charges following well-defined trajectories. The principle of least action is invoked for the derivation of the field equations. The averaging process and its physical implications are not considered in detail. No discussion is included of the difficulties involved in the treatment of the quadratic quantities such as the body forces on the medium, the energy flux, and the elimination of the self-forces in the polarization of the medium. Quantum-mechanical modifications of the theory are not considered.

E. L. Hill (Minneapolis, Minn.).

Lüst, R., und Schlüter, A. Kraftfreie Magnetfelder. Z. Astrophys. **34**, 263–282 (1954).

A force-free magnetic field is defined as one in which the Lorentz force $\operatorname{curl}\mathbf{H}\times\mathbf{H}=0$. This requires that $\operatorname{curl}\mathbf{H}=\alpha\mathbf{H}$, where α is a function of position. The case when $\mathbf{H}$ has symmetry about an axis and α is a constant is considered in some detail. [The authors fail to notice that for the case considered by them, an explicit solution can be found. Thus $\mathbf{H}$ must be of the form

$$\mathbf{H}=\alpha\mathbf{r}\times\mathbf{k}P+\operatorname{curl}(\mathbf{r}\times\mathbf{k}P),$$

where $\mathbf{k}$ is a unit vector in the direction of the axis of symmetry and P satisfies the five-dimensional wave equation

$$\Big(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}-\frac{4\mu}{r^2}\frac{\partial}{\partial\mu}+\alpha^2\Big)P=0.$$

Consequently the general solution for P is a superposition of the fundamental solutions

$$\frac{J_{\pm(n+3/2)}(\alpha r)}{(\alpha r)^{3/2}}C_n^{3/2}(\mu),$$

where $C_n^{3/2}(\mu)$ are the Gegenbauer polynomials (see a forthcoming paper by the reviewer in Proc. Nat. Acad. Sci.)].

S. Chandrasekhar (Williams Bay, Wis.).

Daboni, Luciano. Capacità elettrostatica di un condensatore sferico con apertura circolare. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89**, 208–217 (1955).

The aim of the paper is to compute approximately the capacity C of a modified spherical condenser in which the interior sphere is replaced by a spherical bowl of given opening. Bounds for C are obtained by a simple lemma formulated in Polya and Szegö, "Isoperimetric inequalities in mathematical physics" [Princeton, 1951, p. 62; MR **13**, 270]. Otherwise the procedure follows much the same line as that of an earlier investigation of the author aiming at the capacity of a cube [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 461–466 (1953); MR **15**, 125].

G. Szegö (Stanford, Calif.).

Sen Sarma, Anjan Kumar. An introduction to algebra of contactor relay circuits. J. Assoc. Appl. Phys. Calcutta Univ. **1**, 72–83 (1955).

Some introductory remarks on the algebra of switching networks.

S. Sherman (Philadelphia, Pa.).

Svoboda, Antonín. The synthesis of relay networks. Stroje na Zpracování Informací **2**, 157–208 (1954). (Czech. Russian and English summaries)

A method is described for the synthesis of relay contact networks whose transmissibilities are prescribed tabularly. The contact network formed may have any number of terminals and prescribed transmissibilities for any number of pairs of said terminals.

From the author's summary.

Svoboda, František. The use of indeterminate two-valued Boolean function for the synthesis of switching circuits. Stroje na Zpracování Informací **2**, 209–244 (1954). (Czech. Russian and English summaries)

Fialkow, Aaron, and Gerst, Irving. Impedance synthesis without mutual coupling. Quart. Appl. Math. **12**, 420–422 (1955).

The synthesis method of Bott and Duffin [J. Appl. Phys. **20**, 816 (1949); MR **12**, 307] obviates the use of mutual coupling required by the Brune realization technique. The critical reduction step in the Bott-Duffin synthesis requires six elements and two reduced functions. Here an alternative method is developed by use of a Wheatstone bridge circuit which requires only five elements instead of six. Substantially equivalent results have been discovered independently by F. Reza [J. Math. Phys. **33**, 194–198 (1954); MR **15**, 1009] and also by R. H. Pantell [Proc. I.R.E. **42**, 861 (1954)]. *R. J. Duffin.*

Nenadál, Zdeněk. Multi-terminal resistive networks for the summation of voltages. Stroje na Zpracování Informací **2**, 303–318 (1954). (Czech. Russian and English summaries)

Chowdhury, Arun Kumar. A circuit for calculating the real part of a network function. J. Assoc. Appl. Phys. Calcutta Univ. **1**, 68–71 (1954).

Popov, M. Nouvelle méthode pour la solution des circuits électriques à éléments localisés. Com. Acad. R. P. Romîne **2**, 609–614 (1952). (Romanian. Russian and French summaries)

Marinescu, M., et Popov, M. Une méthode nouvelle pour l'analyse des phénomènes paramétriques. Com. Acad. R. P. Romîne **2**, 615–627 (1952). (Romanian. Russian and French summaries)

Marinescu, Matei, et Popov, Mihai. Sur le spectre des courants dans les circuits en série à inductance variable. Com. Acad. R. P. Romîne **3**, 327–335 (1953). (Romanian. Russian and French summaries)

Marinescu, Matei, et Popov, Mihai. Les relations énergétiques dans le spectre des courants des circuits en série à inductance variable. Le travail mécanique asynchrone. Com. Acad. R. P. Romîne **3**, 337–340 (1953). (Romanian. Russian and French summaries)

Marinescu, M., et Popov, M. Représentation du circuit en série à inductance variable sous forme de réseau aux paramètres constants. Com. Acad. R. P. Romîne **4**, 243–250 (1954). (Romanian. Russian and French summaries)

Quantum Mechanics

Bohm, D., Schiller, R., and Tiomno, J. A causal interpretation of the Pauli equation. A. Nuovo Cimento (10) **1**, supplemento, 48–66 (1955).

This paper brings a contribution to the causal interpretation of nonrelativistic quantum mechanics by means of a hydrodynamic model, as originally proposed by Madelung [Z. Physik. **40**, 322–326 (1926)]. A similar interpretation is here given for the Pauli theory of nonrelativistic particles with spin $\hbar/2$. It makes use of a fluid of little bodies, each of which is rotating around one of its principal axes with angular momentum $\hbar/2$. The bodies can also be pictured as point dipoles of moment $\hbar/2$. The equations of motion couple the angular momentum to the translation of the bodies. The equivalence of the model with the Pauli equation as well as a number of its physical properties are exhibited. *L. Van Hove.*

Bohm, D., and Schiller, R. A causal interpretation of the Pauli equation. B. Nuovo Cimento (10) **1**, supplemento, 67–91 (1955).

The hydrodynamic model proposed in the previous paper for the Pauli theory of nonrelativistic particles of spin $\hbar/2$ is here extended to a fluid of spinning bodies with arbitrary direction of the angular momentum with respect to the axes of the bodies. The non-linear system thus obtained reduces to the Pauli case when certain frequencies are sufficiently low. The case of a hydrogen atom is treated as an example and it is shown how, in the model, the quantization of the total angular momentum of the fluid results from single-valuedness considerations on the field of body orientations. A preliminary discussion is given for the measurement process of the spin. It is emphasized that the observable spin is a collective property of the fluid and is not to be confused with the angular momentum of the individual bodies. The paper concludes with a number of remarks on the possibility of approaching, through causal models for quantum mechanics, a description of a "sub quantum-mechanical level" which might lead to an understanding of elementary particles. *L. Van Hove* (Utrecht).

Destouches, Jean-Louis. L'onde u et le fluide associé dans la théorie de la double solution de M. Louis de Broglie. J. Phys. Radium (8) **16**, 81–85 (1955).

This gives formal, hydrodynamic interpretations of de Broglie's solution of a non-linear wave-equation [C. R. Acad. Sci. Paris **183**, 447–448 (1926); **184**, 273–274; **185**, 380–382 (1927)]. *C. Strachan* (Aberdeen).

Destouches, Jean-Louis. Sur la compatibilité de certaines hypothèses de la théorie de la double solution de M. Louis de Broglie. J. Phys. Radium (8) **16**, 86–91 (1955).

On solutions of a non-linear wave-equation de Broglie has imposed certain conditions; these and their compatibility are discussed. *C. Strachan* (Aberdeen).

de Broglie, L. Une interprétation nouvelle de la mécanique ondulatoire est-elle possible? Nuovo Cimento (10) **1**, 37–50 (1955).

Percus, J. K. Supplementary conditions in quantized linear systems. Phys. Rev. (2) **97**, 1406–1413 (1955).

Hashimoto, Yoshiaki. Note on the commutation relations in quantum mechanics. J. Fac. Sci. Hokkaido Univ. Ser. II. **4**, 209–211 (1952).

The quantum-mechanical equations of motion

$$\dot{q}=[H, q]=p,\ \dot{p}=[H, p]=-\partial V/\partial q$$

do not imply the commutation relation $[p, q]=1$ for $V=0, q^2$. Putnam [Phys. Rev. (2) **83**, 1047–1048 (1951); MR **13**, 1009] has shown that they do for $V=aq^n+b$ (a, b constant, n an odd integer). By an extension of Putnam's argument it is shown that this is not necessarily true when n is even.

C. Strachan (Aberdeen).

Haag, R. On quantum field theories. Danske Vid. Selsk. Mat.-Fys. Medd. **29**, no. 12, 37 pp. (1955).

This is a general survey of the present state of knowledge of the mathematical structure of quantum field theories. Some results are proved by simple and rather abstract arguments, which before were known only under more special conditions.

The point of view is to consider the field-operators $\psi(x)$ and the conjugate momenta $\pi(x)$ of a field theory as a matrix representation of the abstract ring defined by the commutation rules.

(1) $[\psi(x), \psi(x')]=[\pi(x), \pi(x')]=0,\ [\psi(x), \pi(x')]=i\delta(x-x')$.

Two such representations are called equivalent if they are related by a unitary transformation, which must belong to the same Hilbert space in which the operators are defined.

Two main points are made. (a) Although for a system with a finite number of degrees of freedom all representations of (1) are equivalent, this is by no means true for field theories. In particular, the trivial scale transformation (wave-function renormalization)

(2) $$\psi'=c\psi,\ \pi'=c^{-1}\pi,$$

where c is a constant, is not an equivalence transformation according to the definition. (b) The so-called "Haag's Theorem." A field theory in which there is interaction between particles, and one in which there is no interaction, cannot belong to equivalent representations of (1). That is to say, the transformation connecting them cannot belong to the Hilbert space of the non-interacting theory. Result (b) is essentially an old theorem of L. Van Hove [Acad. Roy. Belg. Bull. Cl. Sci. (5) **37**, 1055–1072 (1951); MR **14**, 117] but is here proved in much greater generality. The proof rests on the fact that in the passage between the two theories there is necessarily a wave-function renormalization, i.e. a transformation of the type (2) with $c\neq 1$.

The meaning of these results is to make even clearer than before the fact that the Hilbert space of ordinary quantum mechanics is too narrow a framework in which to give consistent definition to the operations of quantum field theory. It is for this reason that attempts to build a rigorous mathematical basis for field theory within the Hilbert-space framework [e.g. K. O. Friedrichs, Mathematical aspects of the quantum theory of fields, Interscience, New York, 1953; MR **15**, 80] always stop short of any non-trivial examples. The question, what kind of enlarged framework would make consistent definitions possible, is the basic unsolved problem of the subject.

F. J. Dyson (Princeton, N.J.).

★ **Bogoljubow, N. N. Zum Problem der Grundgleichung der relativistischen Quantentheorie der Felder.** Abhandlungen aus der Sowjetischen Physik, Folge IV, pp. 7–10. Verlag Kultur und Fortschritt, Berlin, 1954.

Translation of Dokl. Akad. Nauk SSSR (N.S.) **81**, 757–760 (1951); MR **13**, 711.

Bogolyubov, N. N., and Parasyuk, O. S. On the theory of multiplication of causal singular functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 25–28 (1955). (Russian)

The purpose of this and two following papers is to establish a precise mathematical meaning for the integrals over products of singular functions which always appear in solutions of the equations of quantum field theory. The method is based on an earlier paper [Bogolyubov, same Dokl. (N.S.) **81**, 1015–1018 (1951); MR **13**, 711]. The singular functions are given a rigorous definition as linear functionals over a wider class of functions than that used to define Schwartzian distributions. This paper consists of a very condensed statement without proof of some complicated algebraic lemmas which are impossible to summarize. The lemmas are concerned with the detailed analysis of the structure of products of singular functions, using Feynman graphs and a classification of terms similar to that of A. Salam [Phys. Rev. (2) **82**, 217–227 (1951); MR **13**, 607]. *F. J. Dyson.*

Bogolyubov, N. N., and Parasyuk, O. S. On the subtractive formalism in multiplication of causal singular functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 429–432 (1955). (Russian)

A statement of two theorems, each occupying 20 lines of print, giving quantitative estimates for the most general type of multiple integral over products of singular functions. A proof of the second theorem is sketched, using lemmas taken from the paper reviewed above.

F. J. Dyson (Princeton, N.J.).

Parasyuk, O. S. On the theory of causal singular functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 643–645 (1955). (Russian)

Using the results of the two papers reviewed above, a rigorous justification is given for the process of regularization in quantum field theory [W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434–444 (1949); MR **11**, 301]. It is proved that a product of regularized singular functions always tends to a well-defined limit as the regularizing masses tend to infinity, provided that the convergence is defined in a sense appropriate to linear functionals rather than ordinary functions. The linear functionals are defined over a class $C(q, r, n)$ of functions of n variables; the class is defined by requiring that the function with all its derivatives up to order q, multiplied by any polynomial up to order p, be bounded.

F. J. Dyson (Princeton, N.J.).

★ **Iwanenko, D., und Grigorjew, W. Die Deutung der Regularisierung in der Quantenelektrodynamik.** Abhandlungen aus der Sowjetischen Physik, Folge IV, pp. 69–73. Verlag Kultur und Fortschritt, Berlin, 1954.

Translation of Ž. Eksper. Teoret. Fiz. **21**, 563–566 (1951); MR **13**, 192.

Dalitz, R. H., and Dyson, F. J. Renormalization in the new Tamm-Dancoff theory of meson-nucleon scattering. Phys. Rev. (2) **99**, 301–314 (1955).

The authors study the renormalizations to be carried

out in order to obtain finite results for meson-nucleon scattering in a modified Tamm-Dancoff approximation recently proposed by Dyson [Phys. Rev. (2) **90**, 994; **91**, 1543–1550 (1953); MR **15**, 768] in order to avoid the divergent vacuum self-energy effect appearing in the ordinary Tamm-Dancoff equations. The integral equation for the modified Tamm-Dancoff amplitudes is derived. The nucleon self-energy effects are eliminated by a suitable renormalization of mass and coupling constant. The meson self-energy is more difficult to handle. After a mass renormalization it is not possible to eliminate the remaining divergencies by a new coupling constant renormalization, and the authors simply omit them. The resulting equation has the unsatisfactory feature of possessing in the modified propagator a physically unreasonable pole. Further renormalizations are still needed for the states of isobaric spin $\frac{1}{2}$, and it is found that an unknown parameter is left in the theory. No numerical results are obtained. A comparison is made with the Bethe-Salpeter equation for the same process.

L. Van Hove (Utrecht).

Taylor, J. C. Singular integral equations in quantum field theory. Nuovo Cimento (10) **1**, 679–687 (1955).

This is a discussion of the difficulties associated with the singular integral equations which have appeared in quantum field theory and an attempt to show how they may be overcome. The author has informed the reviewer that the discussion is inaccurate, because of an insufficiently precise use of the limiting processes involving a cut-off and that some of the detailed results are wrong. A correct treatment of a singular integral equation in the new Tamm-Dancoff theory of meson-nucleon scattering has since been published in the paper reviewed above. *A. Salam* (Cambridge, England).

★**Tjablikow, S. W. Adiabatische Form der Störungstheorie im Problem der Wechselwirkung eines Teilchens mit einem gequantelten Feld.** Abhandlungen aus der Sowjetischen Physik, Folge IV, pp. 54–68. Verlag Kultur und Fortschritt, Berlin, 1954.

Translation of Ž. Eksper. Teoret. Fiz. **21**, 377–388 (1951) [MR **13**, 412; cf. **16**, 315].

Polivanov, M. K. On a new derivation of the equations for Green's functions in quantum electrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1061–1063 (1955). (Russian)

The Green's function equations of J. Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37**, 452–455, 455–459 (1951); MR **13**, 520] are here derived simply from the Lagrangian formalism of quantum electrodynamics. The author acknowledges the independent publication of the same derivation by J. L. Anderson [Phys. Rev. (2) **94**, 703–711 (1954); MR **15**, 917]. *F. J. Dyson* (Princeton, N.J.).

Gupta, K. K. The Green's functions for equations of particles of arbitrary spin. Proc. Indian Acad. Sci. Sect. A. **41**, 231–238 (1955).

The author studies the Green's functions associated with a general Bhabha wave equation [Rev. Mod. Phys. **21**, 451–462 (1949); MR **11**, 764]. He gives explicit formulae for the analogues of the Dirac spin-one-half singular functions $S(x)$, $S(x)$, $S_F(x)$. The results for the general case are expressed in terms of the masses and α^k matrices associated with the general wave equation, and the standard singular functions for a spin zero (Klein-Gordon) wave equation. *A. S. Wightman* (Princeton, N.J.).

Markov, M. On the theory of the dynamically deformable form-factor. Dokl. Akad. Nauk SSSR (N.S.) **101**, 51–54 (1955). (Russian)

This is a very general and preliminary discussion of methods of making interactions between elementary particles non-local, without introducing rigid structures defined by fixed form-factors. The proposal is to make the "form-factor" itself a dynamical part of the system, reacting to external forces in a causal way. A set of equations partially embodying this idea is presented.

F. J. Dyson (Princeton, N.J.).

Kurdgelaidze, D. F. Nonlinear scattering in electrodynamics and mesodynamics. Vestnik Moskov. Univ. **9**, no. 8, 81–90 (1954). (Russian)

Three distinct problems are here treated. (a) The scattering of light by light in consequence of the electromagnetic interaction of a charged spin-zero field. Only the low-frequency limit is computed. The result is representable as a non-linear addition to the Lagrangian of the Maxwell field

$$L'=[e^4\hbar/m^4c^7](360\pi^2)^{-1}[a(E^2-H^2)^2+b(E\cdot H)^2]$$

with $a\approx 0.4$, $b\approx 1.5$. (b) The scattering of mesons by mesons in consequence of a scalar or pseudoscalar meson-nucleon interaction. The result is an addition to the meson field Lagrangian

$$L'=-[g^4/\hbar^3c^3](16\pi^2)^{-1}(\log)\phi^4,$$

where (log) represents a logarithmically divergent integral. (c) A discussion of the classical solutions of the non-linear equation for the meson field $(\Box-k_0^2-\lambda\varphi^2)\varphi=0$. Plane wave solutions are obtained in the form of Jacobian elliptic functions, for example

$$\varphi=\varphi_0\,\mathrm{cn}\,(\omega t-k\cdot r+c),\quad \omega^2=k^2+k_0^2+\lambda\varphi_0^2,$$

where the modulus of the elliptic function is given by

$$k_1=[\tfrac{1}{2}\lambda\varphi_0^2/(k_0^2+\lambda\varphi_0^2)]^{\frac{1}{2}}.$$

F. J. Dyson (Princeton, N.J.).

★**Flügge, S. Theoretische Behandlung von Problemen der Mesonenphysik.** Ergebnisse der exakten Naturwissenschaften, Bd. 28, pp. 145–231 (1954). Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

Expository paper.

Petráš, Milan. A contribution to the theory of the Pauli-Fierz equation for a particle with spin 3/2. Czechoslovak J. Phys. **5**, 160–170 (1955). (Russian summary)

The Pauli-Fierz equation for particles with spin $\frac{3}{2}$ [M. Fierz and W. Pauli, Proc. Roy. Soc. London, Ser. A. **173**, 211–232 (1939); MR **1**, 190] is studied by means of a new set of matrices introduced following an idea of Harish-Chandra [ibid. **192**, 195–218 (1948); MR **9**, 557]. Their use is illustrated on the calculation of the magnetic moment.

L. Van Hove (Utrecht).

Bogolyubov, N. N., and Zubarev, D. N. The wave function of the lowest state of a system of interacting Bose particles. Ž. Eksper. Teoret. Fiz. **28**, 129–139 (1955). (Russian)

A simple and physically transparent method is developed for studying the well-known problem of a gas of Bose particles with interaction. Let there be N particles with positions r_j. The quantities

$$(1)\qquad \varrho_k=N^{-\frac{1}{2}}\sum_1^N \exp(-ik\cdot r_j)$$

are then a redundant set of coordinates for the description of the system. If the wave-function is written

$$\varphi=\varphi(t, \varrho_k), \tag{2}$$

then the Schrödinger equation reduces to a partial differential equation for φ in the variables t, ϱ_k. Any solution of this equation, after substituting for the ϱ_k from Eq. (1), will give a correct wave-function for the system in terms of the coordinates r_j. Therefore the redundance of the ϱ_k need not be taken into account in looking for solutions of the Schrödinger equation. By this simple observation, the authors avoid difficulties which similar treatments of the same problem using redundant coordinates have previously encountered [S. Tomonaga, Progr. Theoret. Phys. **5**, 544–569 (1950); MR **13**, 414; D. Bohm and D. Pines, Phys. Rev. (2) **92**, 609–625 (1953); and earlier papers there quoted].

Applying their method in detail to determine the form of the ground state, the authors reproduce rather closely the results of Bohm and Pines. The energy of the system appears as the sum of energies of independent harmonic oscillators (phonons) plus correction terms which couple these oscillators together. In the case of a weak interaction the correction terms are handled by a consistent perturbation theory; the ground-state wave-function is calculated explicitly as far as terms of second order in the perturbation. A previous calculation by A. Bijl [Physica **7**, 869–886 (1940)] is criticized on the ground that some terms which are not small were neglected.

For the low-lying excited states of the system, the wave-functions obtained in the zero-order approximation are

$$\Phi_k=a_k \sum_1^N \exp(ik\cdot r_j)\Phi_0,$$

where Φ_0 is the ground-state. This is identical with the wave-function derived from intuitive arguments by R. P. Feynman [Phys. Rev. (2) **94**, 262–277 (1954)] to represent elementary excitations in liquid helium. The author's theory applied to the liquid helium problem will therefore reproduce the results of Feynman's theory, with the added possibility of proceeding systematically to better approximations and improved wave-functions. *F. J. Dyson.*

Höhler, G. Wechselwirkung eines nichtrelativistischen Teilchens mit einem skalaren Feld für mittlere Kopplung I. Z. Physik **140**, 192–214 (1955).

The non-relativistic motion of a 'polaron', i.e. an electron coupled to, and polarising, a polar crystal lattice is discussed by means of a variation principle. It is assumed that the lowest energy value belongs to a state whose total wave-number (particle+lattice) is zero. Trial functions of a type used by Lee, Low and Pines [Phys. Rev. (2) **90**, 297–302 (1953)] and Pekar [see Fröhlich, Advances in Physics **3**, 325–361 (1954)] and also Gurari [Phil. Mag. (7) **44**, 329-336 (1953)] are discussed. For strong coupling between electron and lattice a new trial function is proposed to account for correlation effects but is not investigated further. Next, a combination of Pekar-type states with different locations of the polaron is used. It generalises, improves on and illumines the trial functions of these other authors. *C. Strachan* (Aberdeen).

Wigner, Eugene P. Lower limit for the energy derivative of the scattering phase shift. Phys. Rev. (2) **98**, 145–147 (1955).

General rules about the energy dependence of scattering phase shifts are given to facilitate the choice between values which fit equally well the scattering cross-section and its angular dependence. If k is the wave number for the incident particle and η the phase shift, $d\eta/dk$ is related physically to the 'retardation' of the particle consequent on scattering. If the scattering centre has 'radius' a, then $d\eta/dk > -a$, essentially. The discussion is given first in terms of a simplified wave-packet formulation for the particle and also by more detailed scattering theory with consequent suggestions about overcoming the difficulty in defining a. *C. Strachan* (Aberdeen).

Schremp, E. J. Isotopic spin and the group space of the proper Lorentz group. Phys. Rev. (2) **99**, 1603 (1955).

Let γ_μ ($\mu=1, 2, 3, 4$) be the usual 4×4 matrices occurring in the Dirac theory of spin one-half particles. Then, with a suitable representation of the γ's, the matrix $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ is diagonal and the four Dirac equations can be written as a pair of coupled 2-component equations [see, e.g., W. Pauli, Handbuch der Physik, 2. Aufl., Bd. 24, Teil 1, Springer, Berlin, 1933, pp. 225–226]. Each solution of these equations is a pair of 2-component spinors. Two such solutions can be associated to form a pair of complex 2×2 matrices. With a proper choice of basis for the two by two matrices the Dirac equations then appear as a pair of quaternion equations. The author proposes to label the first and second columns of his 2×2 matrices by the values ±1 of an isotopic spin variable. It is asserted that in this framework "··· both spin and isotopic spin appear intrinsically···". The motivation for this statement is in earlier work in which the group manifold of the Lorentz group was given a physical interpretation in terms of particle motions. [Only a brief summary of the earlier work has appeared: E. J. Schremp, Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. I, Amer. Math. Soc., Providence, R.I., 1952, pp. 654–655; Phys. Rev. (2) **85**, 721 (1952).] *A. S. Wightman.*

Rayski, J. On a group-theoretical systematization of elementary particles. Bull. Acad. Polon. Sci. Cl. III. **3**, 255–257 (1955).

This is a short note proposing a classification of the elementary particles according to spin, parity and whether they are described by a first- or second-order wave equation. It is asserted that, with appropriate choice of couplings, long life times, copious production of pairs of heavy particles, and the hyperon contribution to nuclear forces can be explained. The scheme is unified by describing all particles in terms of a bilocal field satisfying conditions which are related to those proposed by Yukawa [Phys. Rev. (2) **77**, 219–226 (1950); MR **11**, 567]. That the mass spectrum predicted by the theory does not agree with experiment is attributed to electromagnetic interaction, which the theory neglects. *A. S. Wightman.*

Barker, W. A., and Glover, F. N. Reduction of relativistic two-particle wave equations to approximate forms. III. Phys. Rev. (2) **99**, 317–324 (1955).

A method for reducing the number of components of a wave function was proposed by L. L. Foldy and S. A. Wouthuysen [Phys. Rev. (2) **78**, 29–36 (1950)] and extended to the two-body problem by Z. V. Chraplyvy [ibid. **91**, 388–391; **92**, 1310–1315 (1953); MR **15**, 382, 1011]. This is applied to a sixteen-component wave equation of the Breit type for two fermions with arbitrary masses, charges and intrinsic magnetic moments. The

contribution of the intrinsic magnetic moments to the energy is calculated in the case of hydrogen and positronium. *N. Rosen* (Haifa).

Phan-Van-Loc. Diffraction des ondes Ψ_n de l'électron de Dirac. Ann. Fac. Sci. Univ. Toulouse (4) **18** (1954), 178–192 (1955).

The author derives an expression for the value of the Dirac wave function at a point in terms of its past values on a two-dimensional surface surrounding the point. This expression, specialized to the case of exponential time dependence, is used to solve several diffraction problems (plane wave incident on circular aperture or infinite slit in opaque screen, spherical wave incident on circular aperture in opaque screen, spherical wave incident on opaque half plane). The approximations and results are closely analogous to those of the well-known Kirchhoff theory of diffraction for the scalar wave equation. *A. S. Wightman* Princeton, N.J.).

Cap, F., and Gröbner, W. New method for the solution of the deuteron problem, and its application to a regular potential. Nuovo Cimento (10) **1**, 1211–1222 (1955).

Arguments are presented in support of non-linear terms in the field equations of the pseudo-scalar meson theory. A proton-neutron interaction potential (the derivation of which from a nonlinear field theory is to be published elsewhere), which has a weak singularity, is applied to the deuteron problem. The equations for the ground state of the deuteron are solved by an iteration method. *N. Rosen* (Haifa).

Green, Alex E. S., and Lee, Kiuck. Energy eigenvalues for a spherical well with an exponentially diffuse boundary. Phys. Rev. (2) **99**, 772–777 (1955).

Thermodynamics, Statistical Mechanics

Hooyman, G. J., de Groot, S. R., and Mazur, P. Transformation properties of the Onsager relations. Physica **21**, 360–366 (1955).

The condition that the entropy production be invariant to linear transformations of thermodynamic fluxes and forces is not sufficient to guarantee the invariance of the Onsager relations. It is shown in this paper that for scalar and vector phenomena, this invariance can be inferred from the invariance of the entropy itself.
G. Newell (Providence, R.I.).

★ **Montroll, E. W., and Newell, G. F. Topics in statistical mechanics of interacting particles.** The Institute for Fluid Dynamics and Applied Mathematics, Lecture series, no. 28. University of Maryland, College Park, Md., [1955]. ii+86 pp. (mimeographed).

This lecture series is devoted to the equilibrium statistical mechanics of large systems of interacting particles, and deals with problems which have been the object of considerable attention in recent years. They give in rather concise and self-contained form an excellent survey of the present status of knowledge concerning these problems.

The series opens with an introductory chapter on the basic equations of equilibrium statistical mechanics and on the definition and role of the distribution functions for pairs, triples, etc. of particles. Chapter II is devoted to one-dimensional systems. These are of interest because their statistical properties can be calculated exactly and are thus suitable for a detailed study; they suffer however from the drawback that the phenomenon of phase transition is absent in one dimension. One first treats the case of a one-dimensional gas with interaction between nearest neighbours only. One-dimensional ferromagnets and binary alloys are considered next and serve as an illustration to expound the important matrix method of calculating partition functions.

Chapter III, entitled "Specific heat of solids", deals mainly with the problem of finding the frequency spectrum of elastic vibrations in crystal lattices. Special attention is given to the analytic singularities which have recently been found to occur generally in the spectrum of two and three-dimensional crystals. The two-dimensional Ising model of a ferromagnet is studied in the next chapter. The high and low temperature expansions are first considered. The exact treatment of the problem is then given by the Onsager-Kaufmann method based on spinor calculus.

Chapter V deals with imperfect gases and the theory ot condensation. The first part is devoted to a theorem of the reviewer [Physica **15**, 951–961 (1949)] on the impossibility of finding loops in the pressure-volume diagrams when the canonical configuration integral is calculated exactly. A two-dimensional model of imperfect gas, the so-called lattice gas, is then studied. On the basis of the results established for the Ising model of ferromagnetism the occurrence of condensation can be rigorously established. The chapter ends with the theory of the virial expansion for the equation of state of imperfect gases.

The lectures, although published in mimeographed form, are quite readable. Unfortunately all references to the literature have been omitted, as well as a few figures in Chapter II. *L. Van Hove* (Utrecht).

Van Hove, Léon. Quantum-mechanical perturbations giving rise to a statistical transport equation. Physica **21**, 517–540 (1955).

A presentation is given of a new approach to an old problem. Pauli [Festschrift zum 60. Geburtstag A. Sommerfelds, Hirzel, Leipzig, 1928, p. 30 ff.] first derived an equation aimed at describing how a quantum-mechanical system approached an equilibrium state under the influence of small perturbations and thus represented an ergodic system. This derivation involved the repeated use of a random-phase approximation after each of a long sequence of time intervals, an approximation which in its original form is virtually impossible to justify. By grouping states together as done by van Kampen [Physica **20**, 603–622 (1954); MR **16**, 322], one can obtain a similar transport equation for the probability distribution over the groups and the random phase approximation seems to be less objectionable.

The important feature of Van Hove's derivation of the transport equation for groups of states is the absence of this objectionable repetition of the random-phase approximation. It is assumed, however, that: (1) the energy levels form a continuum (or effectively so) as is true of any physically realistic system of large size; (2) the initial state of the system is one satisfying any of several possible conditions including as a relatively trivial case that in which the random-phase condition is satisfied initially; (3) the perturbing potential λV is considered for sufficiently small values of parameter λ and, what is most important, the matrix or operator V satisfies the

following very special set of conditions which apparently have never previously been considered in this connection but which seem to be valid for most perturbations that might be considered as the cause of ergodic behavior, for example, surface interaction between macroscopic regions, interaction between phonons, etc.

If $E\alpha$ describes an initial state of energy E and other quantum numbers α and if A is any operator diagonal in the representation in which unperturbed Hamiltonian is diagonal, the matrix or kernal $\langle E'\alpha'|VAV|E\alpha\rangle$ is assumed to contain a δ-function singularity of the type $\delta(E'-E)\delta(\alpha'-\alpha)W_A(E\alpha)$ with $W_A(E\alpha)$ not identically zero.

This does seem to be a reasonable assumption. Suppose, for example, that A is the identity operator so that the matrix elements in question describe the usual perturbation scheme for second-order transitions from $E\alpha$ to $E'\alpha'$. If V describes the interaction with a quantized field of some sort, it is likely to involve creation or annihilation operators allowing for the creation or annihilation of only finitely many quanta in each stage. Given any pair of states $E\alpha \neq E'\alpha'$, there is expected to be only a finite number of second-order transition schemes, if any at all, that connect the states $E\alpha$ and $E'\alpha'$. If, however, $E\alpha=E'\alpha'$, any one of an infinite number of possible quanta might be created in the first transition and the same quantum annihilated in the second transition, bringing the system back to its initial state (the self-energy type transitions that plague quantum electrodynamics). Thus there are finitely many transition schemes for $E\alpha \neq E'\alpha'$, but infinitely many for $E\alpha=E'\alpha'$.

Although this reviewer could find no flaws in the derivation, the paper undoubtedly will and indeed should be examined carefully by those who have labored over this very critical problem in the foundations of statistical mechanics. The notation used is essentially that of Dirac [The principles of quantum mechanics, 3rd ed., Oxford, 1947; MR 9, 319] a feature that some people will find annoying. *G. Newell* (Providence, R.I.).

Stratonovič, R. L. Entropy in quantum statistics. Ž. Eksper. Teoret. Fiz. **28**, 547–558 (1955). (Russian)

Consideration of state and entropy of quantum-mechanical systems for nonorthogonal realizations of the wave function. It is shown that from the general formulas follow, on the one hand, the formulas of classical statistics and, on the other hand, known results of quantum statistics. *Author's summary.*

See also: Hooymans, Mazur, and de Groot, p. 102.

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